Improved bounds and new techniques for Davenport–Schinzel sequences and their generalizations

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Definitions.

- Davenport–Schinzel sequence of order s ... *sequence that contains*
  - *no alternation* $a \dots b \dots a \dots$ *of length* $s+2$ *for any pair of symbols* a, b
  - *and no immediately repeated symbol (that is, no* aa*).*
- $\lambda_s(n)$ ... *maximum length of a Davenport–Schinzel sequence of order* s *on* n *distinct symbols (considered as a function of* n*)*
- block ... *contiguous substring with only distinct symbols*
- $\psi_s(m, n)$ ... *maximum length of a Davenport–Schinzel sequence of order* s *on* n *distinct symbols that can be partitioned into* m *or fewer contiguous blocks*
- $\mathrm{ADS}_k^s(m)$-sequence ... *sequence that satisfies:*
  - *It contains no alternation* $abab\dots$ *of length* $s+2$.
  - *It is a concatenation of* m *blocks.*
  - *Each symbol appears at least* k *times (so we have* $m \geq k$*).*
- $\Pi_k^s(m)$ ... *maximum number of distinct symbols in an* $\mathrm{ADS}_k^s(m)$*-sequence*

Observation. $\lambda_1(n) = n$ *and* $\lambda_2(n) = 2n - 1$

Lemma 1 (Agarwal, Sharir, and Shor 1989)**.** *Let* $\varphi_{s-2}(n)$ *be a nondecreasing function in* n *such that* $\lambda_{s-2}(n) \leq n\varphi_{s-2}(n)$ *for all* n*. Then*

$$\lambda_s(n) \leq \varphi_{s-2}(n) \cdot \big(\psi_s(2n, n) + 2n\big).$$

Lemma 2. *For all* s*,* n*,* m*, and* k *we have*

$$\psi_s(m, n) \leq k\big(\Pi_k^s(m) + n\big).$$

Theorem 1 (Klazar 1999)**.** $\lambda_3(n) \leq 2n\alpha(n) + O\big(n\sqrt{\alpha(n)}\big)$

Theorem 2. $\lambda_3(n) \geq 2n\alpha(n) - O(n)$.

Theorem 3. *Let* $s \geq 3$ *be fixed, and let* $t = \lfloor (s-2)/2 \rfloor$*. Then*

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(1/t!)\alpha(n)^t + O\left(\alpha(n)^{t-1}\right)}, & s \text{ even;} \\ n \cdot 2^{(1/t!)\alpha(n)^t \log_2 \alpha(n) + O\left(\alpha(n)^t\right)}, & s \text{ odd.} \end{cases}$$

Theorem 4 (Agarwal, Sharir, and Shor 1989)**.** *Let* $t = \lfloor (s-2)/2 \rfloor$*. Then,*

$$\lambda_s(n) \geq n \cdot 2^{(1/t!)\alpha(n)^t - O\left(\alpha(n)^{t-1}\right)}, \quad s \geq 4 \text{ even.}$$

New proof of Theorem 1.

Lemma 3. *For all* $s \geq 1$*,* $m \geq s$ *we have* $\Pi_s^s(m) = \infty$.

Lemma 4. $\Pi_2^1(m) = m - 1$

Lemma 5. *For all* $s \geq 2$ *we have* $\Pi_{s+1}^s(m) \leq \binom{m-2}{s-1} = O\big(m^{s-1}\big)$.

Recurrence 1. *For every* $s \geq 3$ *and every* k *and* m *we have*

$$\Pi_{2k-1}^s(2m) \leq 2\Pi_{2k-1}^s(m) + 2\Pi_k^{s-1}(m).$$

Corollary 6. *For every fixed $s \geq 2$, if we let $k = 2^{s-1}+1$, then*

$$\Pi_k^s(m) = O\big(m(\log m)^{s-2}\big)$$

(where the constant implicit in the O notation might depend on s).

Recurrence 2. *Let t be an integer parameter, with $t \leq \sqrt{m}$. Then,*

$$\Pi_k^3(m) \leq \left(1+\frac{m}{t}\right)\Pi_k^3(t) + \Pi_{k-2}^3\left(1+\frac{m}{t}\right) + 3m.$$

Corollary 7. *There exists an absolute constant c such that, for every $k \geq 2$, we have*

$$\Pi_{2k+1}^3(m) \leq cm\alpha_k(m) \qquad \text{for all } m.$$

Proof. Let m_0 be a large enough constant and $\widehat{\alpha}_k(x)$, $k \geq 2$, be given by $\widehat{\alpha}_2(x) = \alpha_2(x) = \lceil \log_2 x \rceil$, and, for $k \geq 3$, by

$$\widehat{\alpha}_k(x) = \begin{cases} 1, & \text{if } x \leq m_0; \\ 1 + \widehat{\alpha}_k\big(3\widehat{\alpha}_{k-1}(x)\big), & \text{otherwise.} \end{cases}$$

There exists a constant c_0 such that $|\widehat{\alpha}_k(x) - \alpha_k(x)| \leq c_0$ for all k and x.

We will prove by induction on $k \geq 2$ that

$$\Pi_{2k+1}^3(m) \leq c_1 m\widehat{\alpha}_k(m) \qquad \text{for all } m.$$

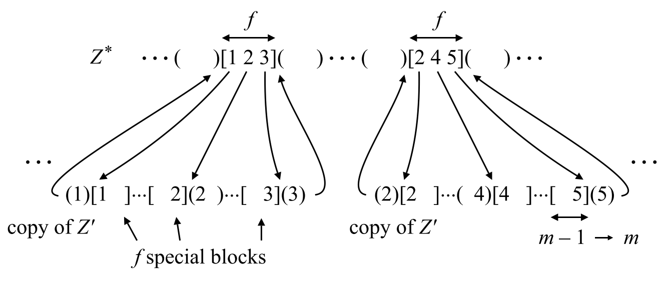
Lemma 8 (Klazar 1999). *We have $\lambda_3(n) \leq \psi_3(1+2n/\ell, n) + 3n\ell$, where $\ell \leq n$ is a free parameter.*

Proof of Theorem 2.

- $Z_d(m)$ ... sequences with the following properties:
  - Each symbol in $Z_d(m)$ appears exactly $2d+1$ times.
  - $Z_d(m)$ contains no *ababa*. (But may contain a repetition.)
  - $Z_d(m)$ is partitioned into blocks. Some of the blocks in $Z_d(m)$ are *special*.
  - Each symbol makes its first and last occurrences in special blocks.
  - Special blocks contain only first and last occurrences of symbols.
  - Each special block in $Z_d(m)$ has length exactly m.
  - For $d \geq 2$, each special block is surrounded by regular blocks on both sides, and *no* regular block is surrounded by special blocks on both sides.
- We enclose regular blocks by ( )'s, and special blocks by [ ]'s.

$$Z_1(m) = [12\dots m](m\dots 21)[12\dots m] \quad Z_d(1) = (\,)[1](1)(1)\dots(1)[1](\,).$$

- $Z' := Z_d(m-1)$.
- $f := S_d(m-1)$ ... the number of special blocks in Z'
- $Z^* := Z_{d-1}(f)$
- $g := S_{d-1}(f)$ ... the number of special blocks in Z^*
- Take one copy of Z^* and g copies of Z', each using its own symbols



- $N_d(m)$ ... number of distinct symbols in $Z_d(m)$
- $V_d(m)$ ... average block length in $Z_d(m)$

Lemma 9. $A_d(m) \leq N_d(m) \leq A_d(m+c)$ *(for $d \geq 3$, $m \geq 2$) and* $V_d(m) \geq m/2$

- Thus $Z_d(d)$ has length $N_d(d)\alpha(N_d(d) - O(N_d(d))$, no *ababa* and removal of repetitions shortens it by at most a $2/N_d(d)$-fraction