A separator theorem for string graphs and its applications

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Definitions.

• String graph *is the intersection graph of continuous arcs in the plane.*

• *Set* $S \subset V(G)$ *is a* separator *in* G *with respect to weights* $w : V(G) \to \mathbb{R}_{\geq 0}$ *(satisfying* $w(V) = 1$*), if there is a partition* $V(G) = S \cup V_1 \cup V_2$ *such that* $w(V_1), w(V_2) \leq 2/3$ *and* $E(V_1, V_2) = \emptyset$*.*

The size of the separator *is the number of its vertices.*

• Bisection width:

$b_w(G) := \min\{|E(V_1, V_2)| : V_1 \cup V_2 = V, w(V_1), w(V_2) \leq 2/3\}$

• $b(G) := b_w(G)$ *with* $w(v) = 1/|V|$

• Pair-crossing number*,* $pcr(G)$*, is the least number of pairs of edges that intersect in a drawing of* G*.*

Theorem 1. *Every weighted string graph has a separator of size* $O(m^{3/4}\sqrt{\log(m)})$*.*

Proof. Separator := vertices of degree at least $m^{1/4}/\sqrt{\log(m)}$ and vertices of the separator from the following lemma. □

Lemma 1. *Every weighted string graph with maximum degree* Δ *has a separator of size* $O(\Delta\sqrt{m}\log(m))$*.*

Proof. string graph $G \to$ topological graph T (i. e., drawn in the plane):

$V(T)$ … one vertex for each edge of G (pair of crossing arcs)

$E(T)$ … parts of the arcs between vertices

Weight of each vertex γ of G is equally distributed among vertices of T lying on γ.

Theorem 2 (Kolman and Matoušek, 2004)**.**

$$b(G) \leq c\log(n)(\sqrt{pcr(G)} + \sqrt{ssqd(G)}),$$

where $ssqd(G) := \sum_{i=1}^{n} deg(v_i)^2$ *and* c *is an absolute constant.*

Corollary 3.

$$b_w(G) = O\left(\sqrt{n}\log(n)\left(\sqrt{dD} + d\right)\right),$$

where d *is the maximum degree and* D *is the maximum number of other edges that an edge crosses.*

Apply Corollary 3 on T.

Separator of G := arcs whose at least one part is in the bisector of T. □

Conjecture (Fox, Pach, Tóth)**.** *Every weighted string graph has a separator of size* $O(\sqrt{m})$*.*

Theorem 4. $\forall \varepsilon > 0 \;\; \exists g(\varepsilon)$ *such that every string graph with girth at least* $g(\varepsilon)$ *has at most* $(1+\varepsilon)n$ *edges.*

Lemma 2 (Pach and Sharir, 2009)**.** $K_{t,t}$*-free string graphs have at most* $n \log(n)^{c_t}$ *edges.*

Lemma 3. *Let* $\alpha > 0$ *and* $\mathcal{F}$ *be a hereditary family of graphs with separators of size* $O(n/\log(n)^{1+\alpha})$*.*

Then $\forall \varepsilon > 0 \;\; \exists g(\varepsilon)$ *such that every graph in* $\mathcal{F}$ *with girth at least* $g(\varepsilon)$ *has at most* $(1+\varepsilon)n$ *edges.*

Lemma 4. *Let* $\varepsilon > 0$*,* g *positive integer and let* $\phi(n)$ *be a decreasing non-negative function satisfying*

$$\phi(g) \leq \frac{1}{12} \quad \text{and} \quad \prod_{i=0}^{\infty}(1+\phi(\lceil (4/3)^i g \rceil)) \leq 1+\varepsilon$$

Graphs from an $n\phi(n)$*-separable hereditary family with girth at least* g *have fewer than* $(1+\varepsilon)n$ *edges.*