Max Cut and the Smallest Eigenvalue

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Theorem 1 (Main) *There is an algorithm that, given a graph $G = (V, E)$ for which the optimum of the Max CUT problem is at least $1 - \varepsilon$, and a parameter δ, finds a vector $y \in \{-1, 0, 1\}^V$ such that*

$$\frac{\sum_{i,j} A_{i,j} |y_i + y_j|}{\sum_i d_i |y_i|} \leq 4\sqrt{\varepsilon} + \delta$$

where $A_{i,j}$ is the weight of edge (i, j) and d_i is the (weighted) degree of vertex i.
The algorithm can be implemented in nearly-linear randomized time $O(\delta^{-2} \cdot (|V| + |E|) \cdot \log |V|)$.

Lemma 2 *If the optimum Max CUT in G has cost at least $1 - \varepsilon$, there is a vector $x \in \mathbb{R}^V$ such that*

$$x^T (D + A) x \leq 2\varepsilon \cdot x^T D x \ .$$

Furthermore, for every $\delta > 0$, we can find in time $O(\delta^{-1} \cdot (|E| + |V|) \cdot \log |V|)$ a vector $x \in \mathbb{R}^V$ such that

$$x^T (D + A) x \leq (2\varepsilon + \delta) \cdot x^T D x$$

Lemma 3 *Given a vector $x \in \mathbb{R}^V$ such that $x^T (D + A) x \leq \varepsilon \cdot x^T D x$, we can find in time $O(|E| + |V| \log |V|)$ a vector $y \in \{-1, 0, 1\}^V$ such that*

$$\frac{\sum_{i,j} A_{i,j} |y_i + y_j|}{\sum_i d_i |y_i|} \leq \sqrt{8\varepsilon} \tag{1}$$