Solving MAX-r-SAT above a Tight Lower Bound

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Definition 1 MAXIMUM r-SATISFIABILITY PROBLEM ABOVE TIGHT LOWER BOUND (MAX-r-SAT$_{\text{TLB}}$)
Input: *Formula F = multiset of m clauses, each with* ***exactly*** *r literals, $k \in \mathbb{N}$*
Question: *Is there a truth assignment satisfying at least $((2^r - 1)m + k)/2^r$ clauses of F?*

We will show: Decideable in $O(m) + 2^{O(k^2)}$-time (r is a fixed constant - thorough the talk)
Some notations:

- $var(C), var(F)$....variables occurring in clause C, in formula F resp.
- $\tau : V \to \{\mathbf{-1}, \mathbf{1}\}$...truth assignment of variables V;
- 2^V ... all truth assignments
- $sat(\tau, F)$...number of clauses of F satisfied by τ
- $sat(F) = \max_{\tau \in 2^{var(F)}} sat(\tau, F)$

Definition 2 Parameterized problem *L is any subset $L \subseteq \Sigma^* \times \mathbb{N}$ (Σ fixed alphabet).*
Parameterized problem L is fixed parameter tractable (FPT) *iff the membership of an instance $(x, k) \in \Sigma^\star \times \mathbb{N}$ in L can be decided in time $|x|^{O(1)} \cdot f(k)$.*
Kernelization *of L is a* ***polynomial time*** *algorithm mapping (x, k) to (x', k') (the* kernel*) s.t.*

(i) $(x, k) \in L \Leftrightarrow (x', k') \in L$

(ii) $k' \leq f(k)$

(iii) $|x'| \leq g(k)$

for some functions f, g. $g(k)$ ... size of the kernel

Fact 1 *$L \in FPT$ iff L decideable and admits kernelization.*

bikernelization from L to L' ... same as kernelization except (i) $(x, k) \in L \Leftrightarrow (x', k') \in L'$

Lemma 2 *If there is a polynomial (size) bikernel from L to L', L is NP-hard, and L' is in NP, then there is a polynomial kernel for L.*

Definition 3 MAX r-LIN2$_{\text{TLB}}$
Input: *m linear equations $e_1, \ldots, e_m$ in n variables over $\mathbb{F}_2$, no equation has more than r variables, $w_j \in \mathbb{N}$ weight of e_j, $k \in \mathbb{N}$*
Question: *Is there an assignment of $\{0, 1\}$ to variables s.t. the total weight of satisfied equations is at least $(W + k)/2$, where $W = \sum w_j$?*

F ... r-CNF formula with clauses $C_1, \ldots, C_m$ and variables $x_1, \ldots, x_n$

Algebraic representation

$$X = \sum_{C \in F} \left[1 - \prod_{x_i \in var(C)} (1 + \epsilon_i x_i) \right], \text{ where } \epsilon_i = \begin{cases} -1 & x_i \in C \\ 1 & \overline{x_i} \in C \end{cases}$$

Lemma 3 $\forall \tau \in 2^{var(F)} : X = 2^r(sat(\tau, F) - (1 - 2^{-r})m.$

Now we can rewrite X as

$$X = \sum_{I \in S} X_I; \quad X_I = c_I \prod_{i \in I} x_i; \quad c_I \in \mathbb{Z} \setminus \{0\}, \quad S \subseteq \binom{\{1, \ldots, n\}}{r}$$

Our question: $\exists x_1, \ldots x_n : X(x_1, \ldots x_n) \geq k$?
we show: S large $\Rightarrow$ answer is yes
we assume: each x_i randomly independently -1 with prob. 1/2 and 1 with prob. 1/2.

Lemma 4 $\forall X$ *real random variable with* $0 < \mathbb{E}(X^4) < \infty$*:*

$$\mathbb{E}(|X|) \geq \frac{\mathbb{E}(X^2)^{3/2}}{\mathbb{E}(X^4)^{1/2}}$$

Corollary 5 X *real rand. var.,* $\mathbb{E}X = 0, \mathbb{E}X^2 = \sigma^2 > 0, \mathbb{E}X^4 \leq b\sigma^4$*. Then* $\mathbb{P}(X \geq \frac{\sigma}{2\sqrt{b}}) > 0$.

Lemma 6 (Bourgain 1980) *Let* $f(x_1, \ldots, x_n)$ *be a real polynomial of degree* r*. Choose* $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ *uniformly at random and set* $X = f(\epsilon_1, \ldots, \epsilon_n)$*. Then* $\mathbb{E}X^4 \leq 2^{6r}(\mathbb{E}X^2)^2$.

Lemma 7 *Let* $X = \sum_{I \in S} X_I$ *as above, assume* $X \not\equiv 0$*. Then* $\mathbb{E}X = 0, \mathbb{E}X^2 = \sum_{I\ inS} c_I^2 \geq |S| > 0, \mathbb{E}X^4 \leq 2^{6r}(\mathbb{E}X^2)^2$.

Theorem 8 (Main theorem) Max-r-Sat$_{\text{TLB}}$ *is FPT, can be solved in time* $O(m) + 2^{O(k^2)}$*. Moreover, there exist an* $O(k^2)$*-size bikernel from* Max-r-Sat$_{\text{TLB}}$ *to* Max-r-Lin2$_{\text{TLB}}$ *and* $O(k^2)$ *kernel for* Max-r-Sat$_{\text{TLB}}$.