Lion and Man – Can Both Win?

B. Bollobás, I. Leader, M. Walters
presented by Tomáš Valla

November 12, 2009

- Motivation(Rado, 1930): Lion and gladiator (man) in the closed unit disc D, same speed. Does the lion catch the man?
- There was a generally believed proof that lion catches the man, but there was error in the proof.

Proposition 1 *(Besicovitch, 1952) The man is able to evade the lion forever.*

- Lion path is a function $l : [0, \infty) \to D$ such that $|l(s) - l(t) \leq |s - t|$ for all s, t (the path is "Lipschitz") and with $l(0) = x_0$ for some fixed $x_0 \in D$. Similarly for man path.
- L is the set of all lion paths, M set of all man paths.
- Strategy for the lion is a function $G : M \to L$ such that if $m, m' \in M$ agree on $[0, t]$ then also $G(m), G(m')$ agree on $[0, t]$. ("no lookahead"). Strategy for the lion is winning if for every $m \in M$ there is time t with $G(m)(t) = m(t)$. Similarly for man.
- The disc D can be replaced by a metric space X.
- Does the lion in the original game have a winning strategy as well? No, but the proof is not straightforward.

Proposition 2 *The lion does not have a winning strategy in the original game.*

- The bounded-time lion and man game in metric spce X is a lion-man game, where there is a fixed parameter $T > 0$, and the lion wins if he has caught the man by time T while the man wins otherwise. For the rest of the talk we consider only this game.
- Let X be a metric space, let $\varepsilon > 0$ be such that $T = n\varepsilon$. In the ε-discrete bounded-time game on X lion and man take turns (say lion first). The player runs for a time ε, the game ends after both players had n turns. The outcome of the game is the closest distance d that occured between the lion and man at any time.
- There is a $\delta = \delta(\varepsilon)$ such that for any $\delta' < \delta$ the man has a strategy that ensures d is at least δ' and for any $\delta' > \delta$ the lion has a strategy that ensures $d < \delta'$.

Theorem 1 *For the bounded-time game on X, with δ and ε as above, we have*

1. *If $\delta \to 0$ as $\varepsilon \to 0$ in the discrete game then the lion wins the continuous game.*
2. *If $\delta \not\to 0$ as $\varepsilon \to 0$ tn the discrete game then the man wins the continuous game.*
3. *If the lion wins the continuous game then $\delta \to 0$ as $\varepsilon \to 0$ in the discrete game.*
4. *If the man wins the continuous game then $\delta \not\to 0$ as $\varepsilon \to 0$ in the discrete game.*

Lemma 1 *Suppose that $\delta \not\to 0$ as $\varepsilon \to 0$. Then the man has a winning strategy in the continuous game.*

Lemma 2 *Suppose that the lion has a winning strategy for the continuous game. Then $\delta \to 0$ as $\varepsilon \to 0$.*

Lemma 3 *Let X be compact. Suppose that for every n the exists a lion strategy G_n in the continuous game such that for every man path m we have $d(G_n(m)(t), m(t)) < 1/n$ for some t. Then there exists a winning lion strategy in the continuous game.*

- A partial strategy is a function G from a subset of M to L that satisfies "no lookahead" where it is defined (if G is defined at $m, m' \in M$ and m, m' agree on $[0, t]$, then also $G(m), G(m')$ agree on $[0, t]$.
- A partial strategy G is good if for each m for which $G(m)$ is defined there is a subsequence of th paths $G_n(m)$ that converges uniformly to $G(m)$.
- Given two good partial strategies G_1, G_2 with domains M_1, M_2, we say $G_1 \leq G_2$ if $M_1 \subseteq M_2$ and $G_2|M_1 = G_1$.

Corollary 1 *Suppose that in the ε-discrete game we have $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then the lion has a winning strategy for the continuous game.*

Theorem 2 *In the bounded-time game played in a compact metric space, at least one of the lion and man has a winning strategy.*

Lemma 4 *Let X be a compact metric space. Suppose that the man has a continuous winning strategy. Then $\delta \not\to 0$ as $\varepsilon \to 0$.*

Theorem 3 *In the lion and man game in the closed unit disc the man does not have a continuous winning strategy. Indeed, for any continuous man strategy there is a lion path catching the man by time 1.*