A Short Proof of the Hajnal-Szemerédi Theorem on Equitable Coloring

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Definition 1 An equitable k-coloring *of a graph $G = (V, E)$ is a proper k-coloring, for which any two color classes differ in size by at most one.*

Theorem 1 *If G is a graph satisfying $\Delta(G) \leq r$ then G has an equitable $(r+1)$-coloring.*

From now on, let G be a graph with $s(r + 1)$ vertices.
Take $G \cup K_p$ for a suitable $p \leq r$ to achieve this.

Definition 2 A nearly equitable $(r + 1)$-coloring *of G is a proper coloring f, whose color classes all have size s except for one* small class *$V^- = V^-(f)$ with size $s - 1$ and one* large class *$V^+ = V^+(f)$ with size $s + 1$.*

Given such a coloring f, define the auxiliary digraph $H = H(G; f)$ as follows: The vertices of H are the color classes of f. A directed edge VW belongs to $E(H)$ iff some vertex $y \in V$ has no neighbors in W. In this case we say that y is movable *to W.*

Call $W \in V(H)$ accessible, *if V^- is reachable from W in H. V^- is trivially accessible. Let $\mathcal{A} = \mathcal{A}(f)$ denote the family of accessible classes, $A := \bigcup \mathcal{A}$ and $B := V(G) \setminus A$.*
Let $m := |\mathcal{A}| - 1$ and $q := r - m$. Thus $|A| = (m + 1)s - 1$ and $|B| = qs + 1$.

Lemma 2 *If G has a nearly equitable $(r + 1)$-coloring f, whose large class V^+ is accessible, then G has an equitable $(r + 1)$-coloring.*

Definition 3 *A class $V \in \mathcal{A}$ is* terminal, *if V^- is reachable from every class $W \in \mathcal{A} \setminus \{V\}$ in the digraph $H \setminus \{V\}$.*
Every non-terminal class W partitions $\mathcal{A} \setminus \{W\}$ into two parts S_W and $T_W \neq \emptyset$, where S_W is the set of classes that can reach V^- in $H \setminus \{W\}$.

Choose a non-terminal class U so that $\mathcal{A}' := T_U \neq \emptyset$ is minimal. Then every class in $\mathcal{A}'$ is terminal and no class in A' has a vertex movable to any class in $(\mathcal{A} \setminus \mathcal{A}') \setminus \{U\}$. Set $t := |\mathcal{A}'|$ and $A' := \bigcup \mathcal{A}'$.

Definition 4 *Call an edge zy with $z \in W \in \mathcal{A}'$ and $y \in B$,* a solo edge *if $N_W(y) = z$. The ends of solo edges are called* solo vertices *and vertices linked by solo edges are called* special neighbors *of each other. Let S_z denote the set of special neighbors of z and S^y denote the set of special neighbors of y in A'.*

Lemma 3 *If there exists $W \in \mathcal{A}'$ such that no solo vertex in W is movable to a class in $\mathcal{A} \setminus \{W\}$ then $q + 1 \leq t$. Furthermore, every vertex $y \in B$ is solo.*

Lemma 4 *If $V^+ \subseteq B$ then there exists a solo vertex $z \in W \in \mathcal{A}'$ such that either z is movable to a class in $\mathcal{A} \setminus \{W\}$ or z has two nonadjacent special neighbors in B.*

Theorem 5 *There exists an algorithm $\mathcal{P}'$ that from input $(G; f)$ constructs an equitable $(r+1)$-coloring of G in $c(q+1)n^3$ steps.*

Theorem 6 *There is an algorithm $\mathcal{P}$ of complexity $O(n^5)$ that constructs an equitable $(r+1)$- coloring of any graph G satisfying $\Delta(G) \leq r$ and $|G| = n$.*

Theorem 7 (Kierstead, Kostochka 2007) *Every graph satisfying $d(x)+d(y) \leq 2r+1$ for every edge xy, has an equitable $(r+1)$-coloring.*

Conjecture 8 (Seymour '73) *Every graph with minimum degree $\delta(G) \geq \frac{k}{k+1}|G|$ contains the k-th power of a hamiltonian cycle.*

Proved for large graphs (in terms of k) by Komlós, Sarkozy and Szemerédi in 1998 using the Regularity Lemma, the Blow-up Lemma and the Hajnal-Szemerédi Theorem.