Small-size ε-Nets for Axis-Parallel Rectangles and Boxes

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Definitions

A *range space* $(X, \mathcal{R})$: $X =$ set of objects (e.g. all points in $\mathbb{R}^2$), $\mathcal{R} \subseteq 2^X =$ collection of ranges (e.g. axis-parallel rectangles)

Given a range space $(X, \mathcal{R})$, a finite subset $P \subset X$, and a parameter $0 < \varepsilon < 1$, an *ε-net* for P and R is a subset $N \subseteq P$ with the property that any range $r \in \mathcal{R}$ with $|r \cap P| \geq \varepsilon |P|$ contains an element of N.

Known results

- for any range space $(X, \mathcal{R})$ with bounded VC-dimension and for any P and ε there is an ε-net of size $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$. In geometry: $X =$ points, $\mathcal{R} =$ "simple" regions (axis-parallel rectangles, fat triangles, discs ...)
- lower bounds in "geometric cases" only $\Omega(\frac{1}{\varepsilon})$
- upper bound $O(\frac{1}{\varepsilon})$ for half-spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$, discs, pseudo-discs

Main theorem

Theorem 1. *For any set P of n points in the plane and a parameter $\varepsilon > 0$, there exists an ε-net for P and axis-parallel rectangles, of size $O(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$.*

Proof

Construction of the net N

- $r := 2/\varepsilon$, $s := cr \log \log r$, for some constant $c > 1$
- construct a balanced binary tree structure $\mathcal{T}$ over P of depth $1 + \log r$
- fix a uniform random sample $R \subseteq P$, each point is taken with probability $\pi := s/n$. Thus $\mathbf{E}[|R|] = s$.
- for each node v of $\mathcal{T}$ we define a strip σ_v, a vertical line ℓ_v, sets $P_v = P \cap \sigma_v$ and $R_v := R \cap \sigma_v$, and a set $\mathcal{M}_v$ of maximal open R-empty axis-parallel rectangles contained in σ_v and attached to the "entry side" of σ_v
- $|\mathcal{M}_v| = 2|R_v| + 1$

- number of maximal R-empty rectangles for any fixed level of $\mathcal{T}$ is $O(|R|+r)$
- for each node v of $\mathcal{T}$ and for each $M \in \mathcal{M}_v$, the *weight factor* $t_M := s \cdot \frac{|M \cap P|}{n}$. M is *heavy* if $t_M \geq s/r = c \log\log r$, i.e., if $|M \cap P| \geq n/r = \frac{\varepsilon}{2} n$.
- for each heavy M, there is a $\frac{1}{t_M}$-net N_M for $M \cap P$ of size $c' t_M \log t_M$
- $N := R \cup \bigcup_{M \text{ heavy}} N_M$

N is indeed an ε-net

It suffices to show for heavy R-empty rectangles Q contained in some strip σ_v attached to its entry side. There is $M \in \mathcal{M}_v$ containing Q and $|Q \cap N_M| \geq 1$.

Estimating the expected size of N

$$\mathbf{E}[|N|] = cr \log\log r + c' \cdot \mathbf{E}\left[\sum_v \sum_{M \in \mathcal{M}_v, t_M \geq c \log\log r} t_M \log t_M\right]$$

- fix a level i of $\mathcal{T}$, define $\mathrm{CT}(R) := \bigcup_{v \text{ at level } i} \mathcal{M}_v$, and $\mathrm{CT}_t(R) := \{M \in \mathrm{CT}(R); t_M \geq t\}$.
- let R' be another random sample of P, where each point is taken with probability $\pi' := \pi/t$
- Exponential decay lemma:
$$\mathbf{E}[\mathrm{CT}_t(R)|] = O(2^{-t})\mathbf{E}[\mathrm{CT}(R')|]$$
- $t := c \log\log r$, so $\pi' = r/n$
- $|\mathrm{CT}(R')| \leq 2|R'| + 2r$, hence $\mathbf{E}[\mathrm{CT}(R')|] = O(r)$.
- $\mathbf{E}[\mathrm{CT}_t(R)|] = O(r/2^{c \log\log r}) = O(r/\log^c r)$, and similarly for any $j \geq t$, $\mathbf{E}[\mathrm{CT}_j(R)|] = O(r/2^j)$
- contribution of the i-th level:
$$\mathbf{E}\left[\sum_{v \text{ at level } i} \sum_{M \in \mathcal{M}_v, t_M \geq t} t_M \log t_M\right] = O\left(\frac{r \log\log r \log\log\log r}{\log^c r}\right)$$
- in total
$$\mathbf{E}[|N|] = O\left(r \log\log r + \frac{r \log\log r \log\log\log r}{\log^{c-1} r}\right) = O(r \log\log r)$$

$\square$