A bipartite strengthening of the Crossing Lemma

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Crossing Lemma: Let $G = (V, E)$ be a graph with n vertices and $m \geq 4n$ edges drawn in the plane. Then G has at least $\Omega(\frac{m^3}{n^2})$ pairs of crossing edges.

BASIC DEFINITIONS:

crossing - pair of curves and common interior point between two arcs.

crossing number $cr(G)$ - minimum number of crossings in a drawing of G

l-grid - is a pair two disjoint edge subset $E_1, E_2 \subset E$ of a drawing of a graph $G = (V, E)$, such that $|E_1| = |E_2| = l$ and every edge in E_1 crosses every edge in E_2

bi-clique - complete bipartite graph where vertex classes differ in size by at most 1

x-monotone curve - a curve that intersect every vertical line in at most one point

x-monotone drawing - drawing of graph G such that every edge is mapped to an x-monotone curve

bisection width $b(G)$ - the smallest nonnegative integer such that there is a partition of the vertex set $V = V_1 \cup^* V_2$ with $\frac{1}{3}|V| \leq V_i \leq \frac{2}{3}|V|$ for $i = 1, 2$, and $|E(V_1, V_2)| = b(G)$.

THEOREMS:

Theorem 1.2 For every $k \in \mathbb{N}$, there is a constant $c_k > 0$ such that every drawing of a graph $G = (V, E)$ with n vertices and $m \geq 3n$ edges in which no two edges cross in more than k points contains an l-grid with $l \geq c_k \frac{m^2}{n^2}$.

Theorem 1.3 For every positive n, there is an x-monotone drawing of the complete bipartite graph $K_{n,n}$ such that $l = O(\frac{n^2}{\log n})$ for every l-grid in the drawing.

Theorem 1.4

(i) Every drawing of a dense graph $G = (V, E)$ with n vertices and $m = \Theta(n^2)$ edges contains an l-grid with $l = \Omega(\frac{n^2}{\log n})$.

(ii) There is a constant c such that every drawing of a graph $G = (V, E)$ with n vertices and $m \geq 3n$ edges contains an l-grid with $l \geq \frac{m^2}{n^2 \log^c(m/n)}$.

Theorem 2.2 Let G be a graph with n vertices of degree $d_1, d_2, \ldots, d_n$. Then

$$b(G) \leq 10\sqrt{cr(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$$

Algorithm 2.3 Decompose(G'):

1. Let $S_0 = \{G'\}$ and $i = 0$
2. While $(3/2)^i < 4n^2/m$, do

   Set $i := i + 1$. Let $S_i := \emptyset$. For every $H \in S_{i-1}$, do

   - If $|V(H)| \leq (2/3)^i 2n$, then let $S_i := S_i \cup \{H\}$;

   - otherwise split H into induced subgraphs H_1 and H_2 along a bisector of size $b(H)$, and let $S_i := S_i \cup \{H_1, H_2\}$.
3. Return S_i