Computing Barriers of Ordinary Differential Equations

Stefan Ratschan

June 9, 2015

Simulation and Interval Methods

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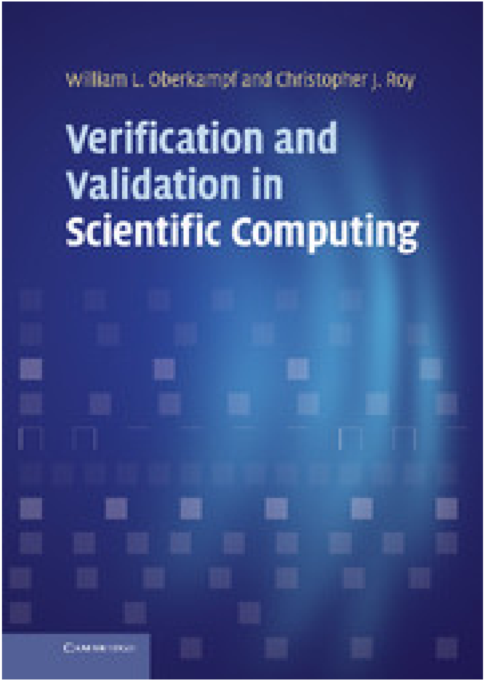
What about verification?


William L. Oberkampf and Christopher J. Roy
Verification and Validation in Scientific Computing

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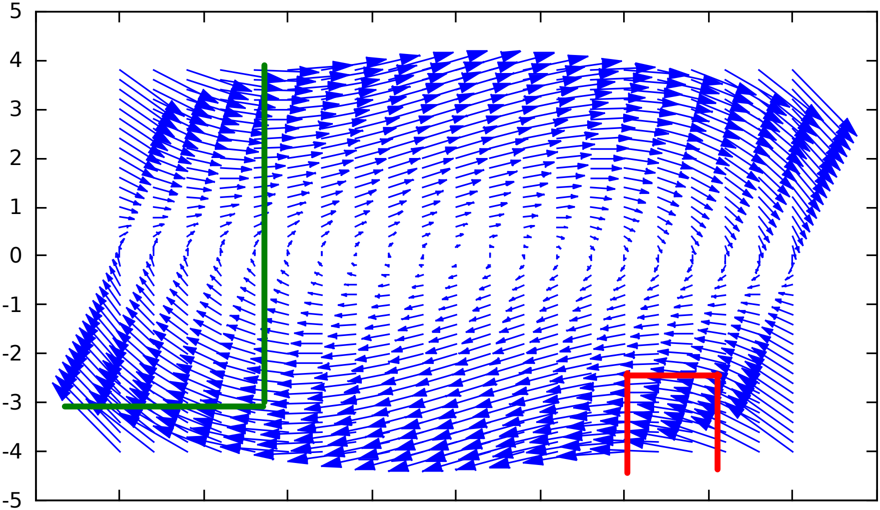
Specific question: safety verification
Does the system always stay in safe range?
. . . never reach an unsafe state?

Example



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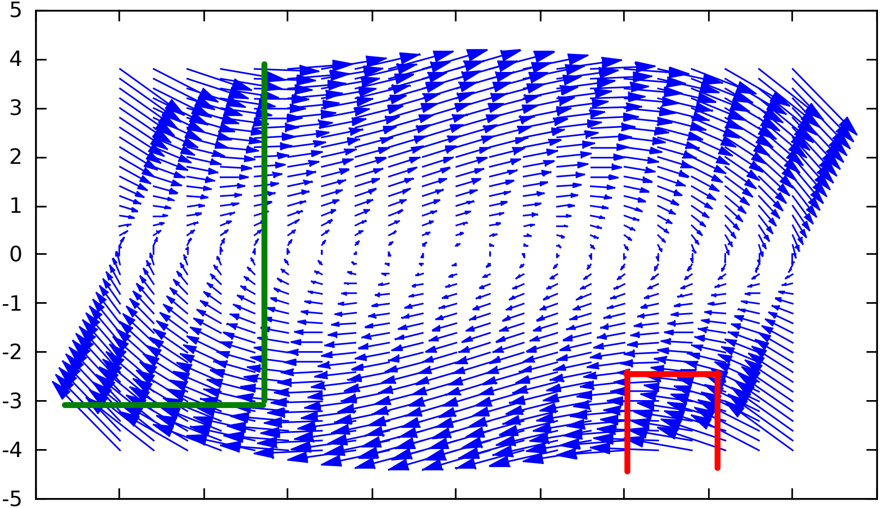

Classical numerical methods cannot exclude this

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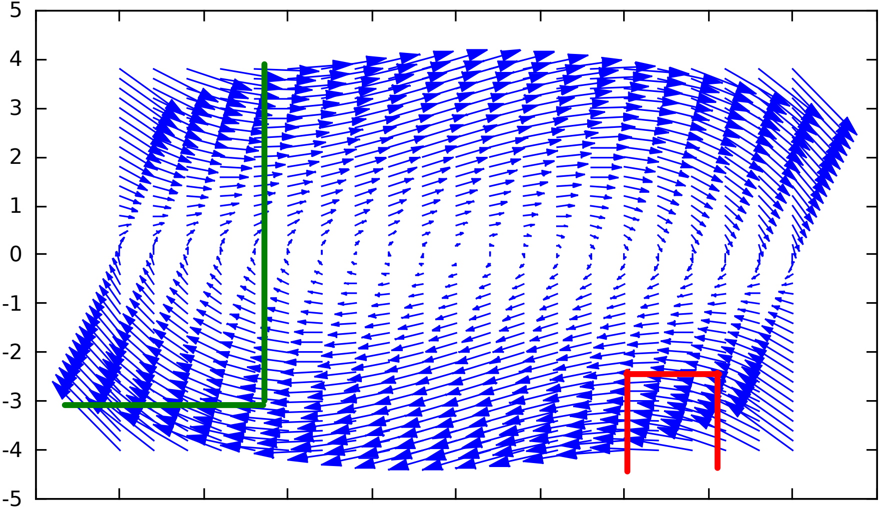

Interval methods will (often) blow up

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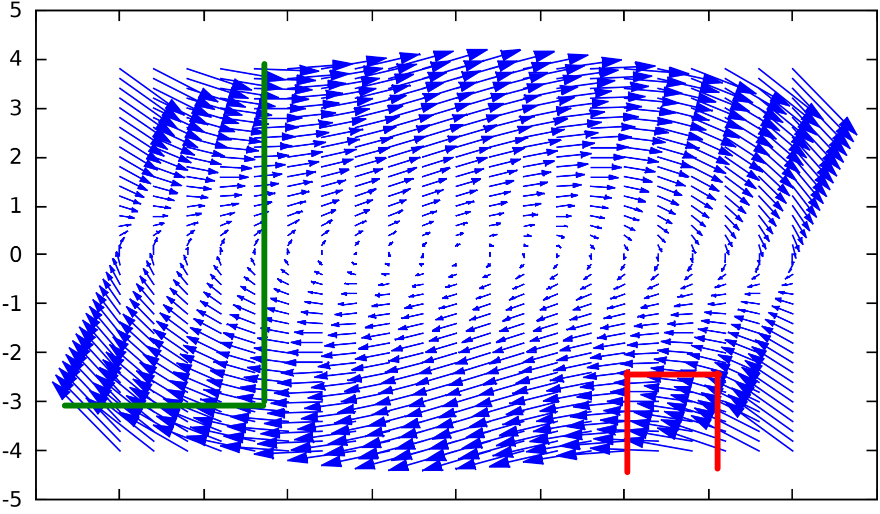

any alternative?

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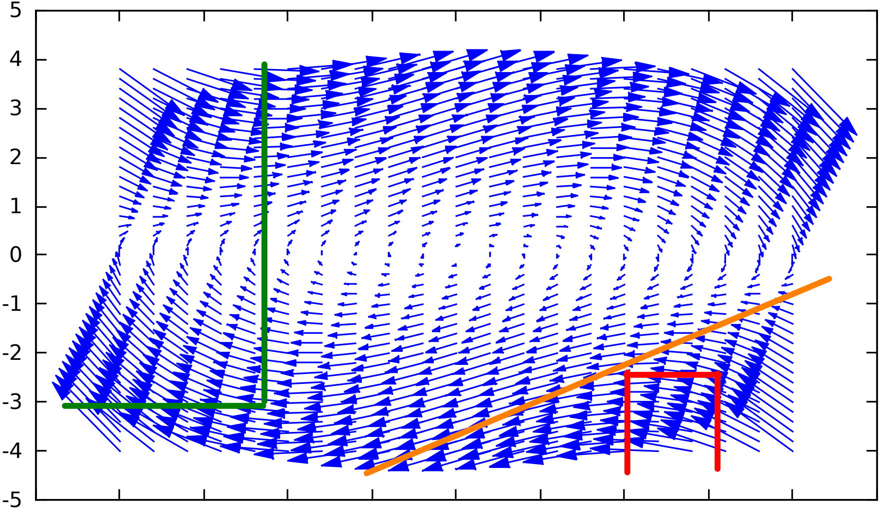

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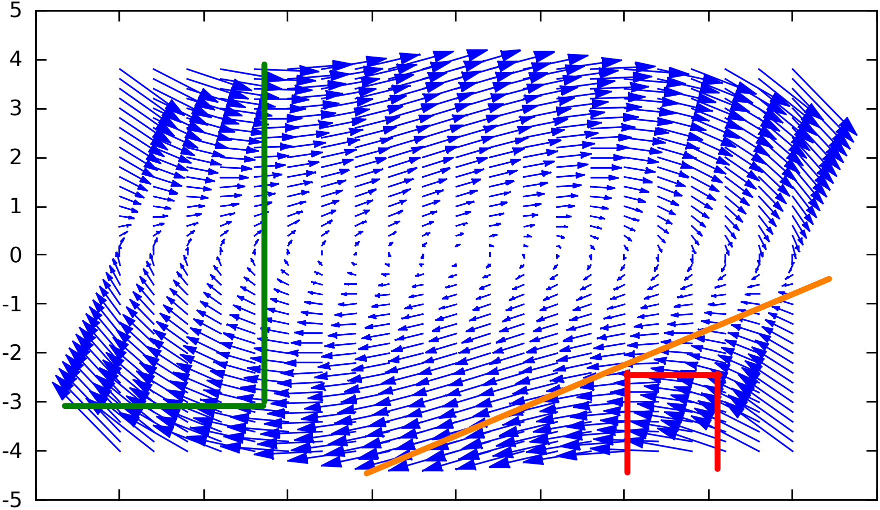

Prajna [2003]

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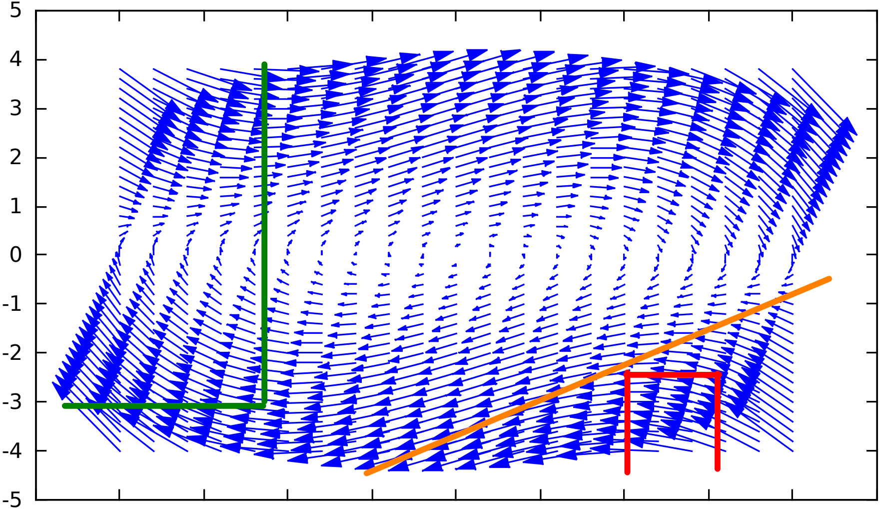
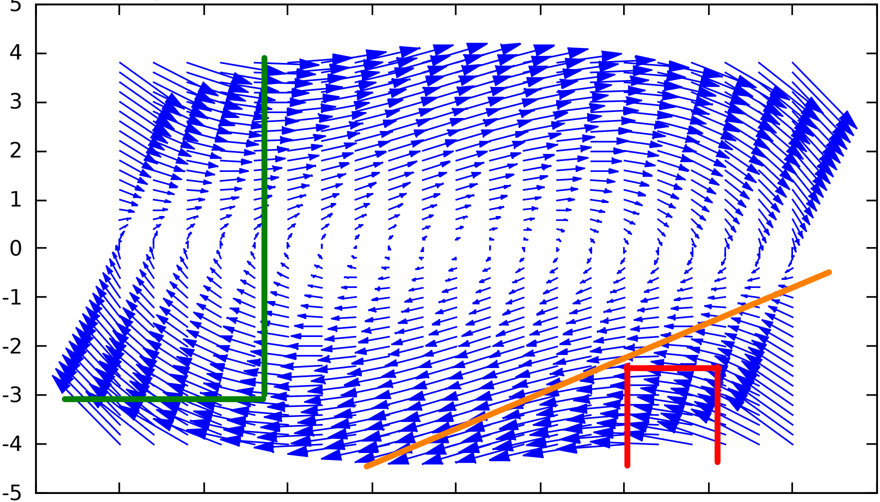


Can this be automatized?

How to Represent the Certificate?



Intuition: function V s.t.

- V is negative on Init, positive on Unsafe
- V decreases along the vector field on $V = 0$

Problem Formalization

Given:

- an n-dimensional ODE $\dot{x} = f(x)$, with
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- $\forall x \in B \,.\, V(x) = 0 \Rightarrow \nabla_f V(x) < 0$, where
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Intuition: parametric function, for example: $ax^2 + bxy + cy^2$.

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How to solve such a problem?

Solving Barrier Conditions

This is a quantified constraint:

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Problem: huge computational complexity

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Prototype implementation: no experiments yet.

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More experiments and development needed . . .

Literature I

Olivier Bouissou, Alexandre Chapoutot, Adel Djaballah, and Michel Kieffer. Computation of parametric barrier functions for dynamical systems using interval analysis. In *IEEE CDC 2014*, pages 753–758, 2014.

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