Non linear Gaussian inversion

With application to robotics mobile mapping

J. Nicola and L. Jaulin, SWIM 2015
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- Context
- Problem
- Classical method
- Proposed method
- Improvements
- Application
- Conclusion

- State estimation

$$\begin{cases} \dot{x} = f(x,u) + \omega_\alpha \\ y = g(x) + \omega_\beta \end{cases}$$

Evolution equation

Observation equation

- Evolution equation
  - Models the system dynamics

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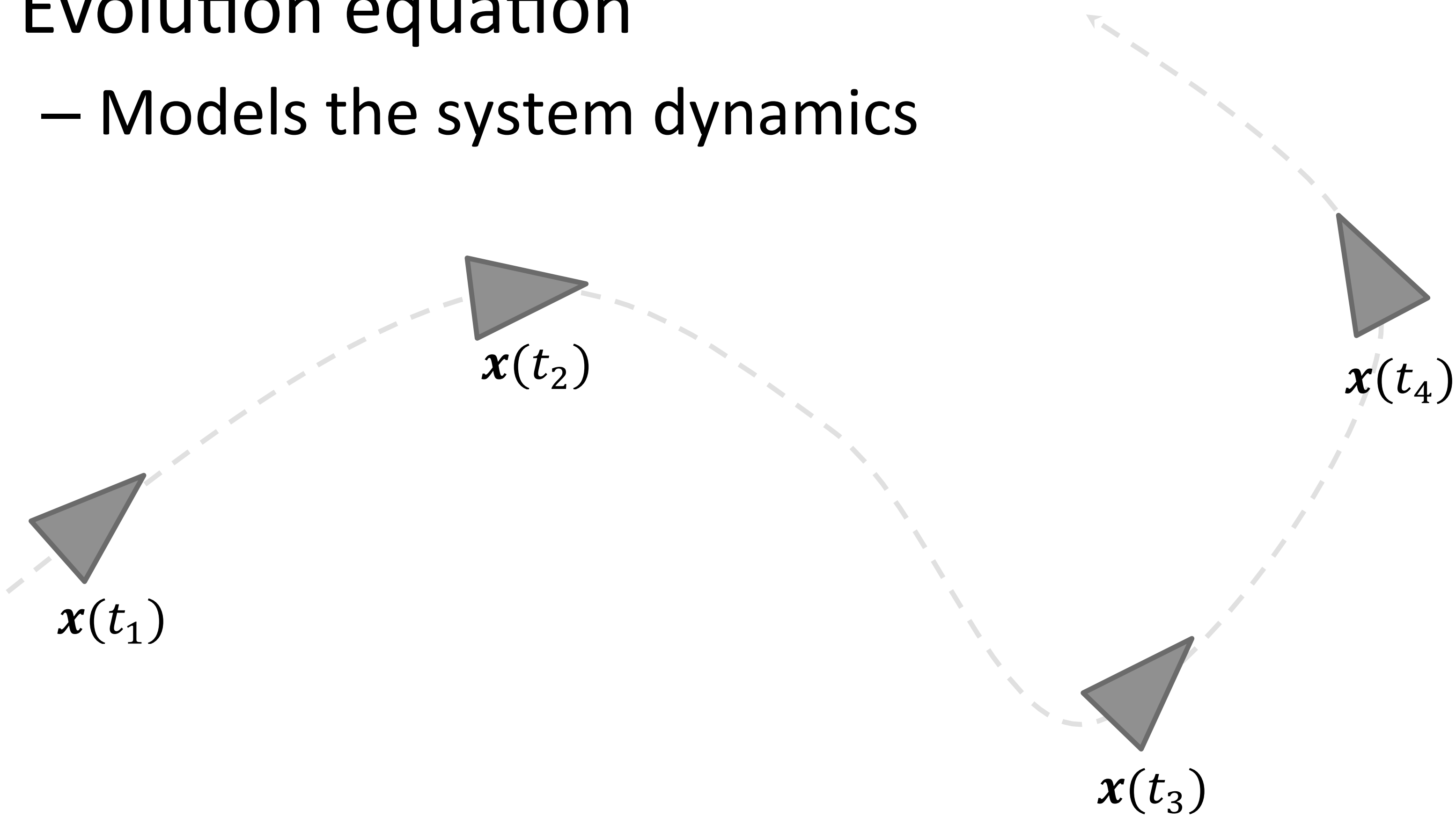
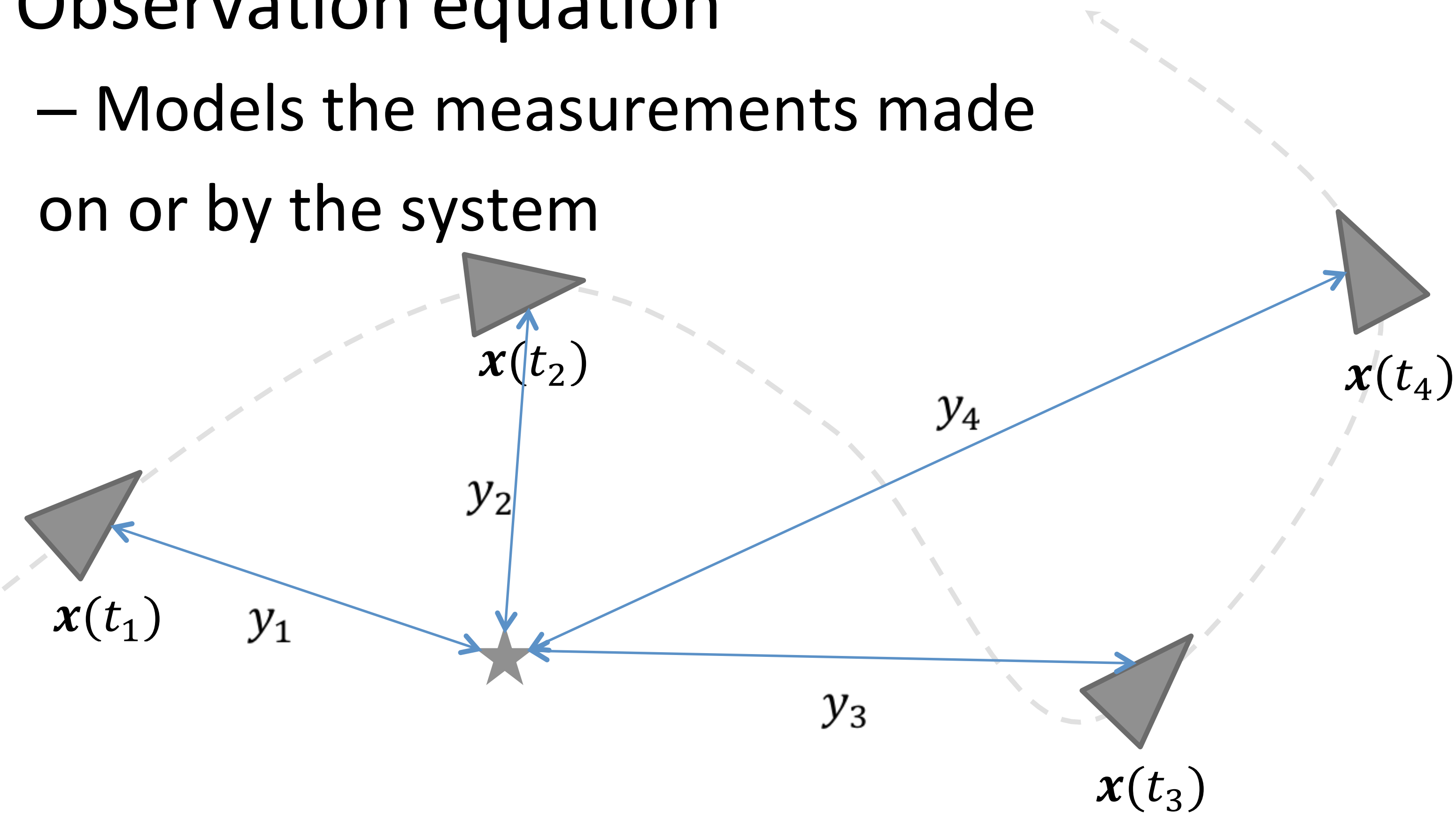


Example: the motion of a robot

- Observation equation
  - Models the measurements made on or by the system



Example: measuring distances to a beacon

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  - Why?
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    - For the evolution equation, see:
      - Kalman Contractor, J. Nicola and L. Jaulin, SWIM 2014

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- Problem: how can we improve the precision of the observation equation?

$$y = g(x) + \omega_\beta$$

We know y (ex: a distance measurement, a GPS position...)

$$y = g(x) + \omega_\beta$$

We know y up to some noise (additive, white, Gaussian)

We have a model for *g* (ex: a distance function)

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We have a model for *g*

We want to find *x*
(ex: the position of the robot)

$$y = g(x) + \omega_\beta$$

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- Example $y = 20e^{-x_1 t} - 8e^{-x_2 t}$

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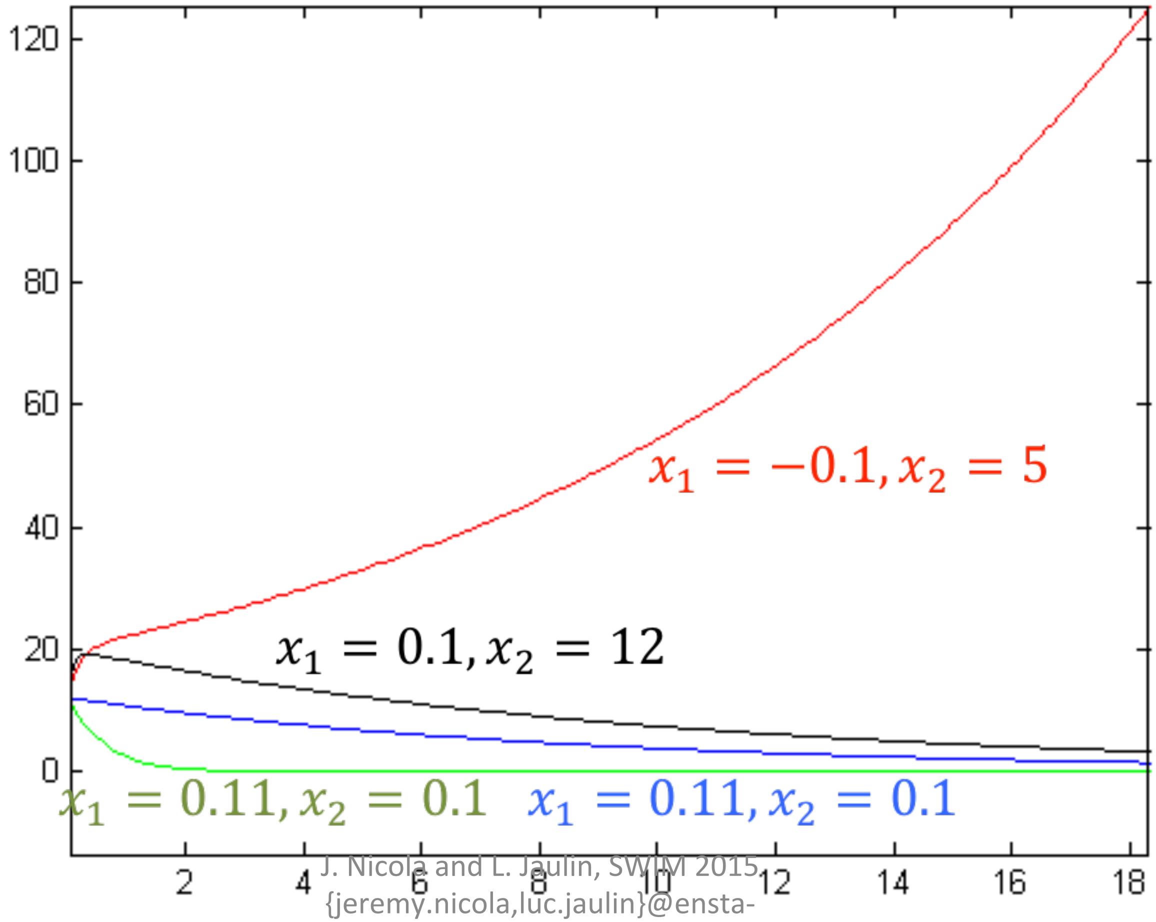

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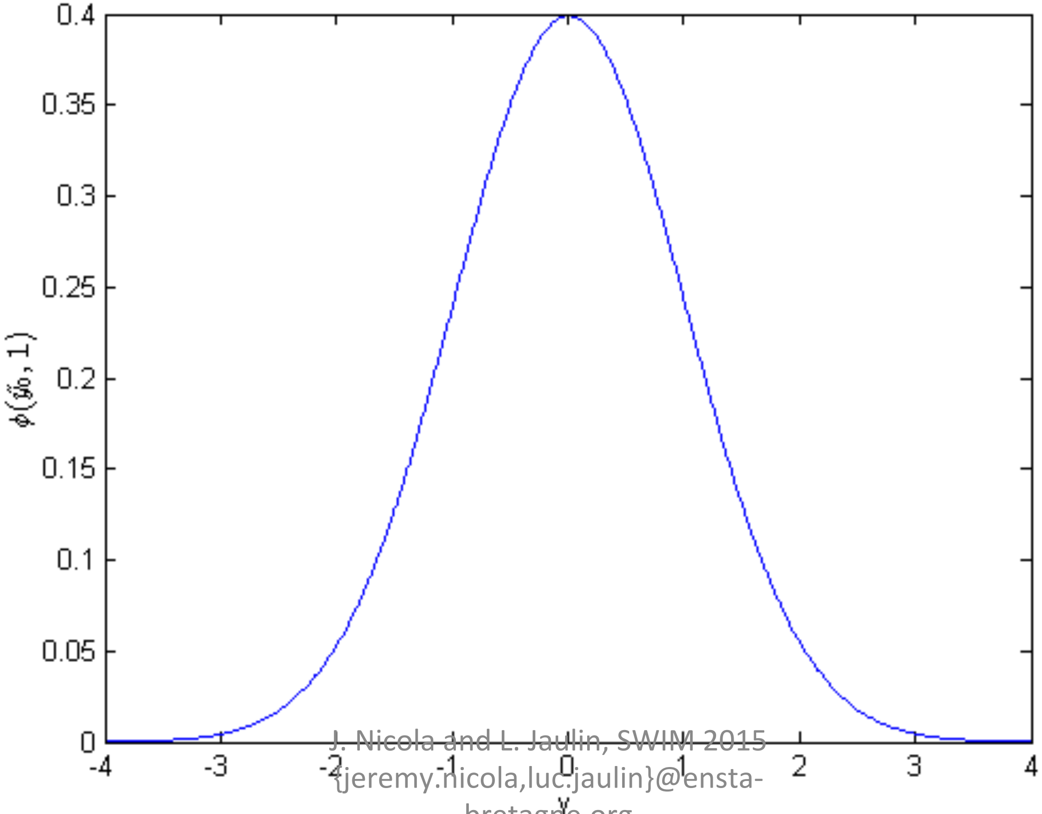

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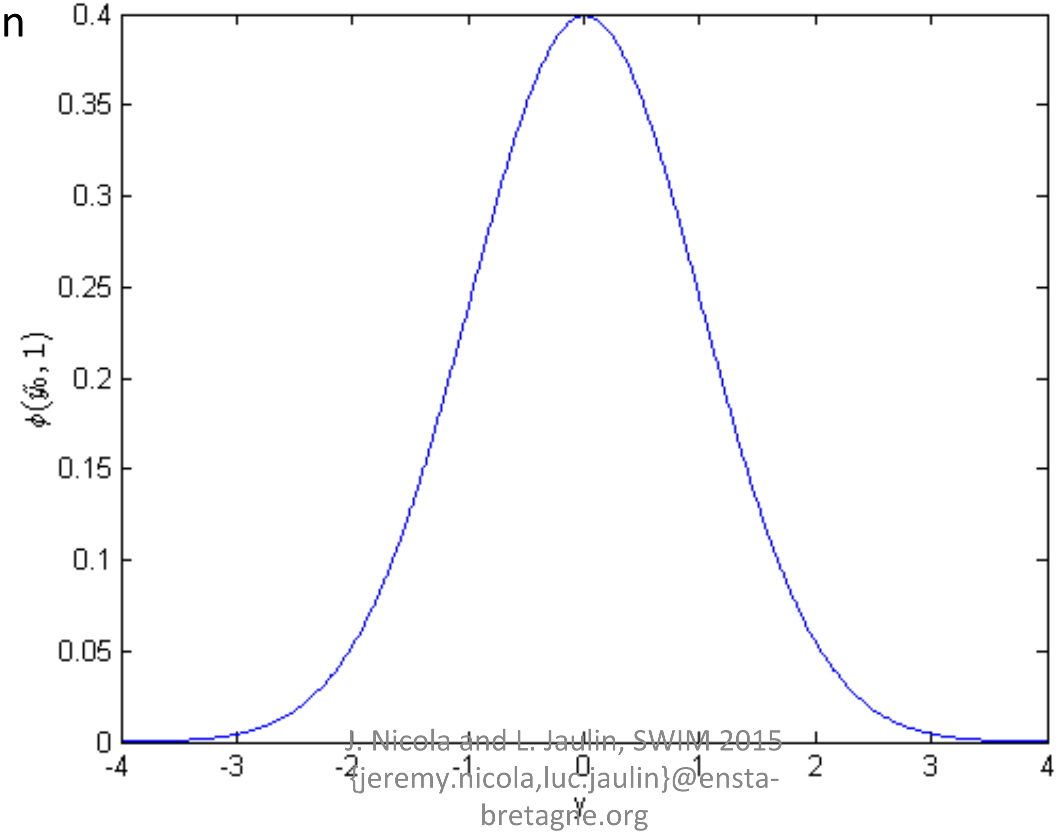
A confidence domain for y is defined by :

$$y^t y < a^2(\eta)$$



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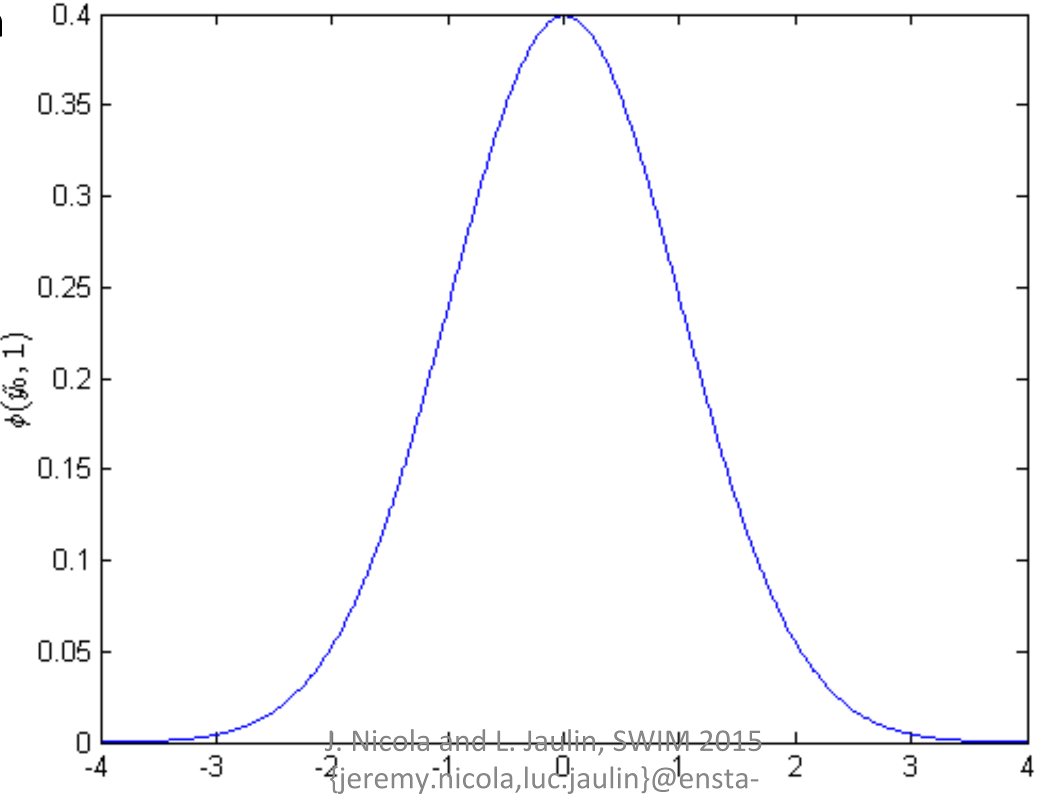

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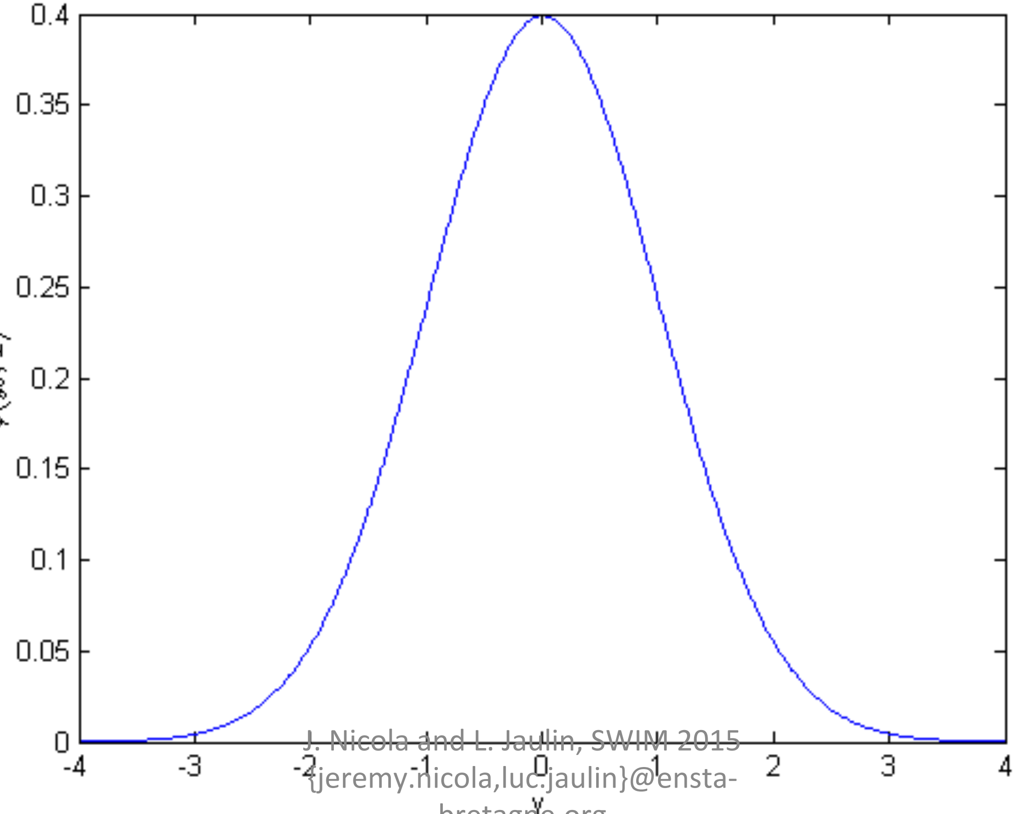

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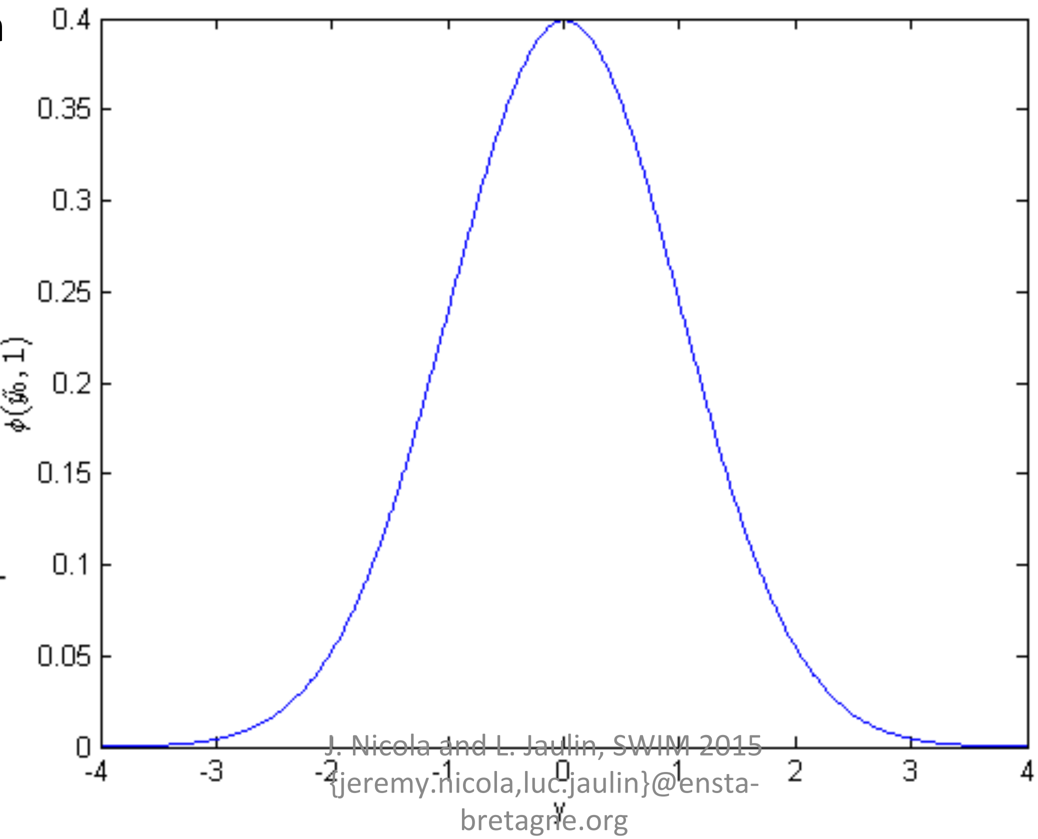

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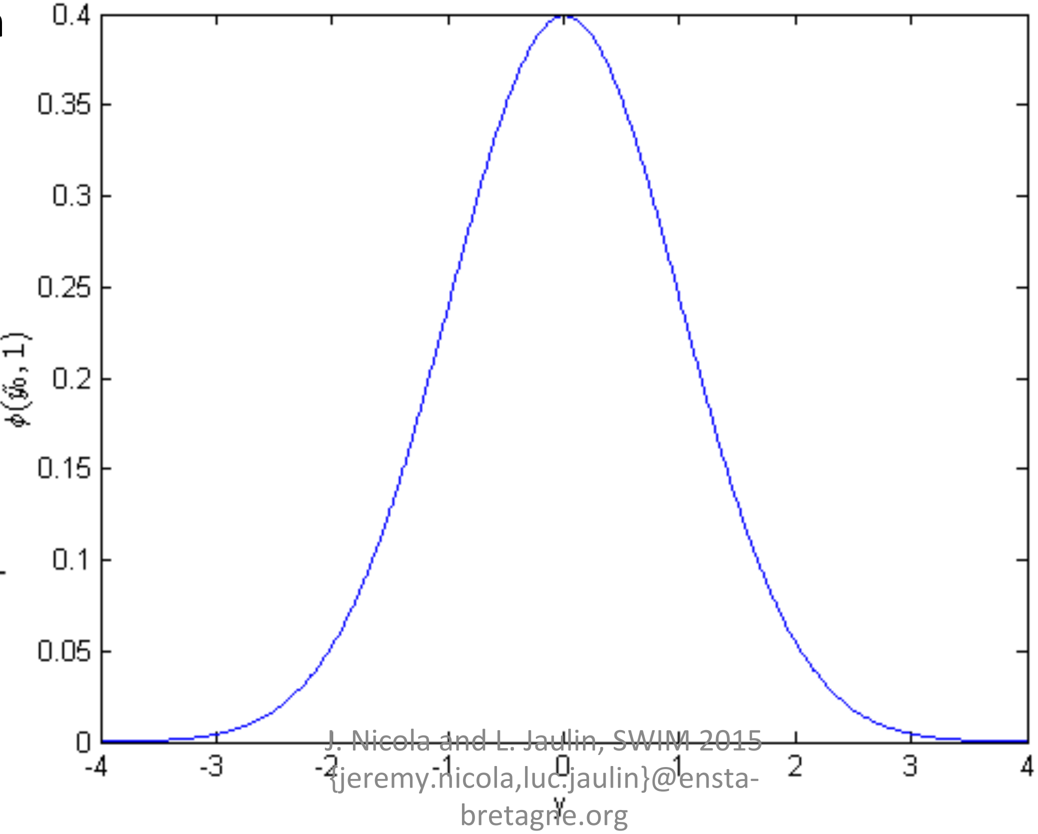

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$$\eta = \int_0^{a^2} \pi(z,k)dz$$

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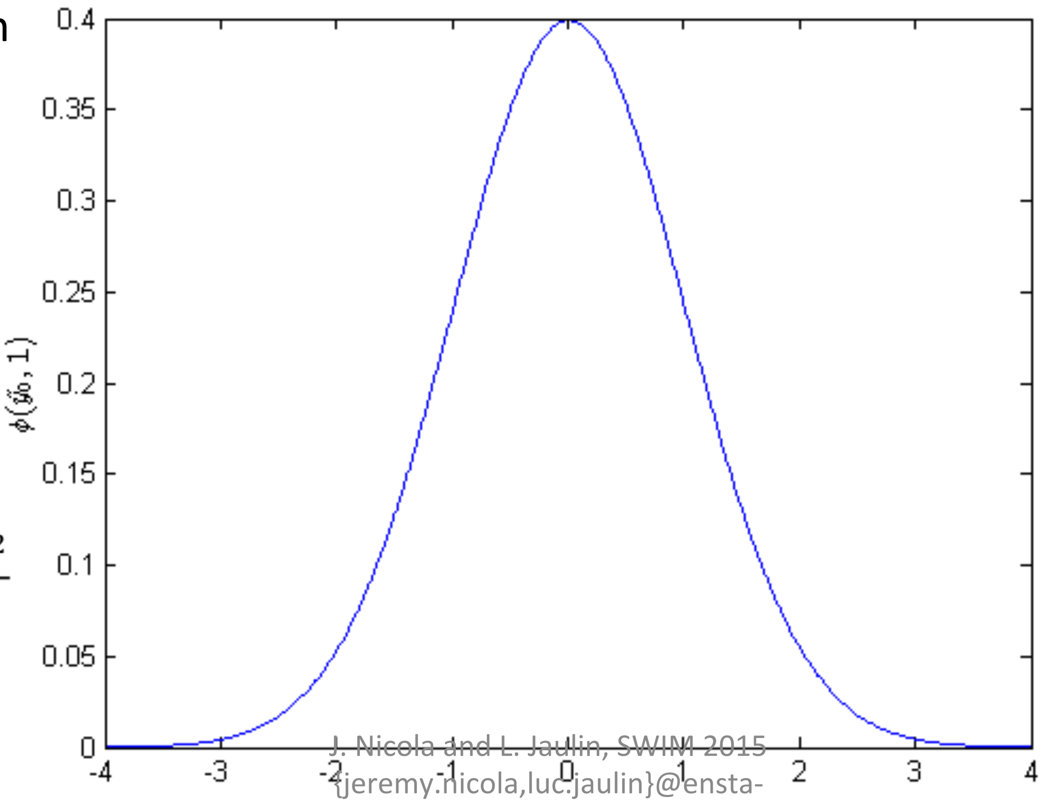

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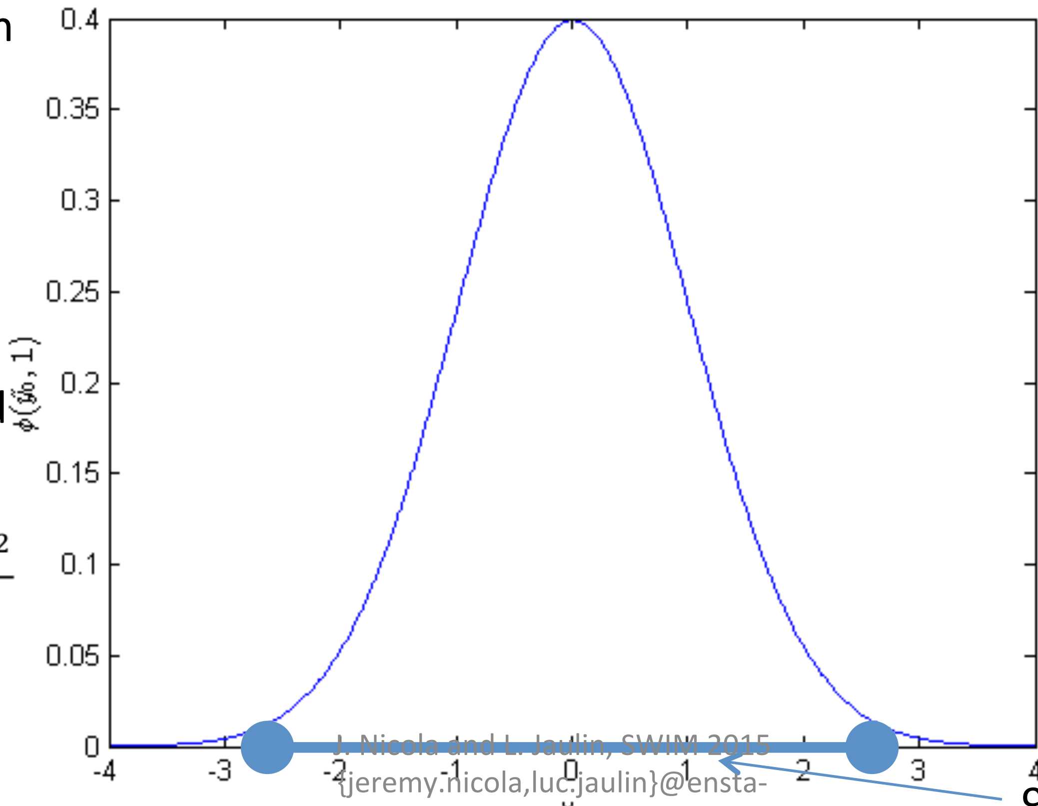

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$$y_0 \in \tilde{y}_0 + [-2.58, 2.58]$$

99% confidence domain

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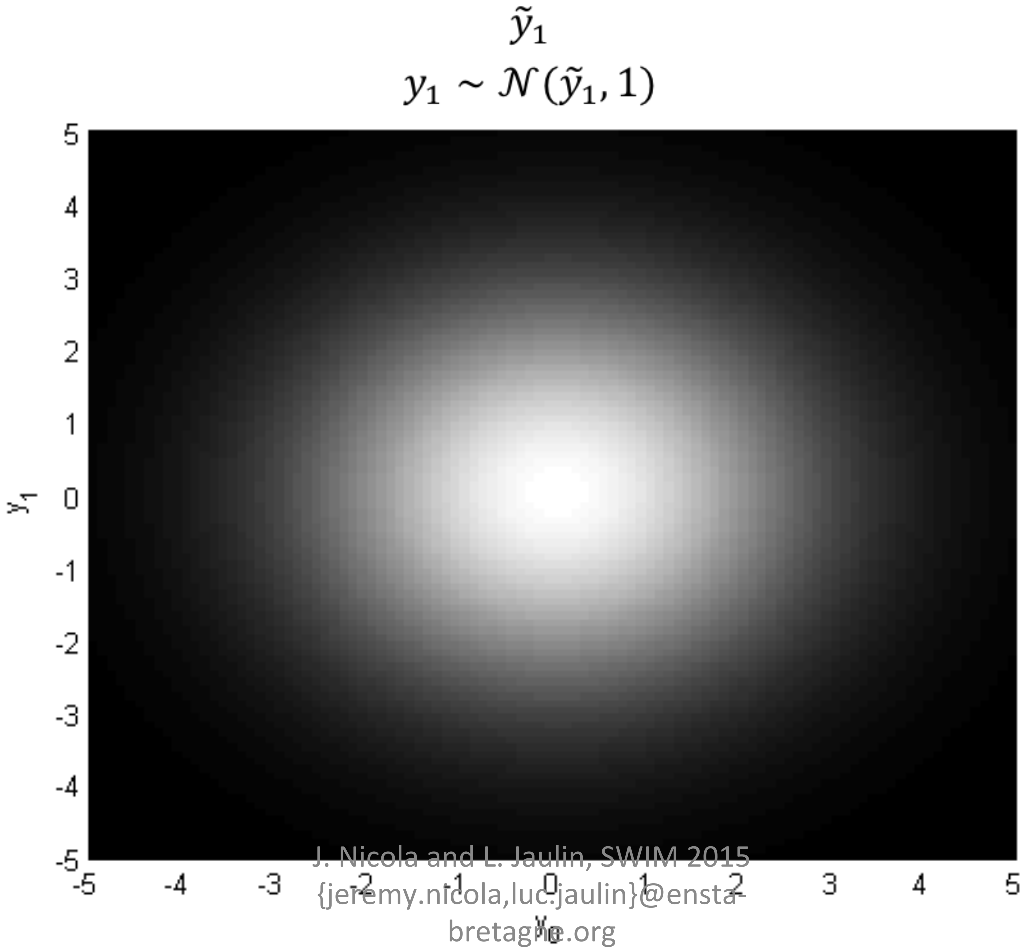

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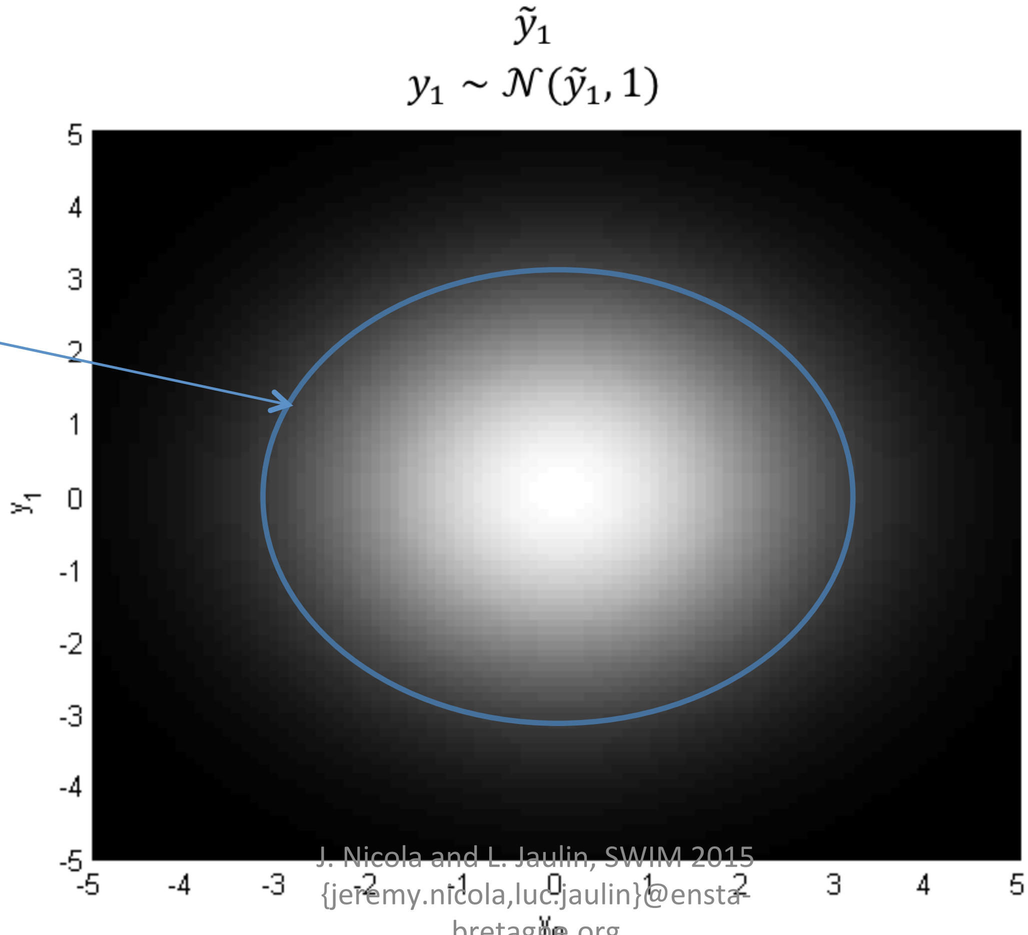
$a(0.99,2)=3,04$

0,99% confidence domain



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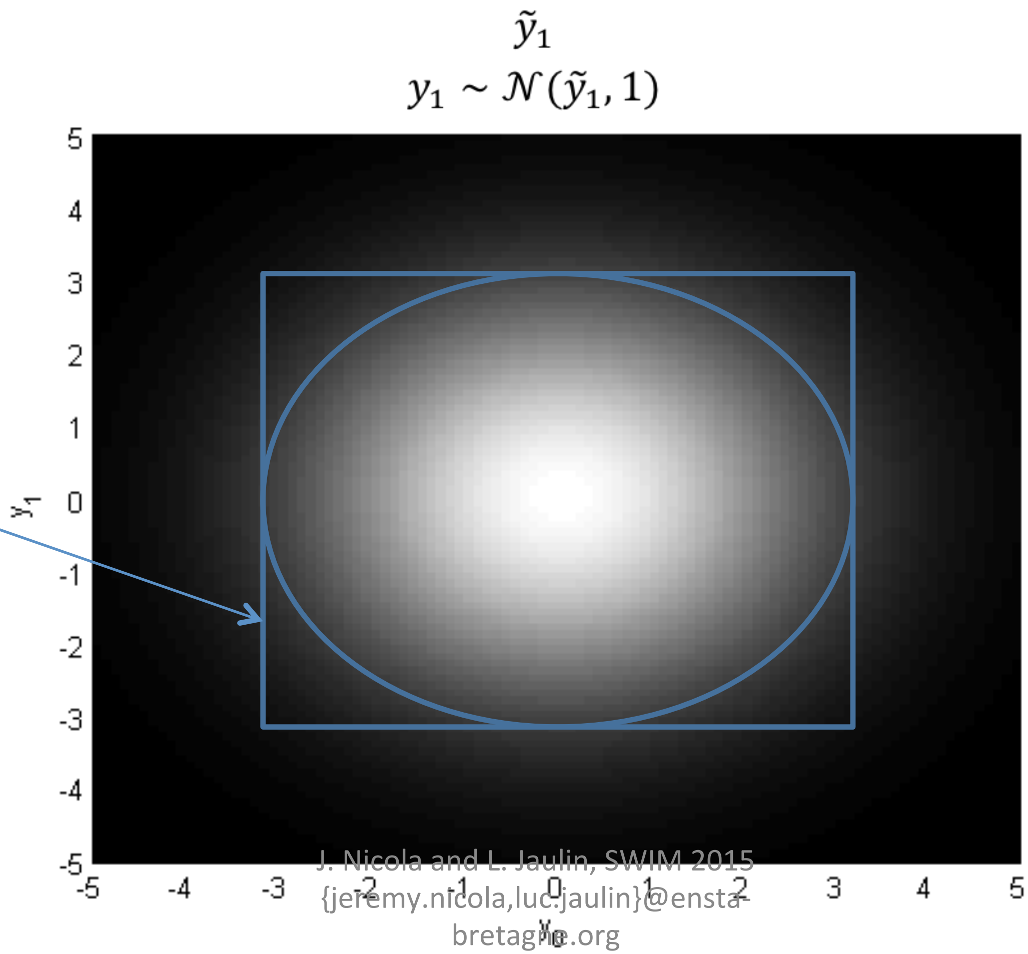
Box-hull of the confidence domain



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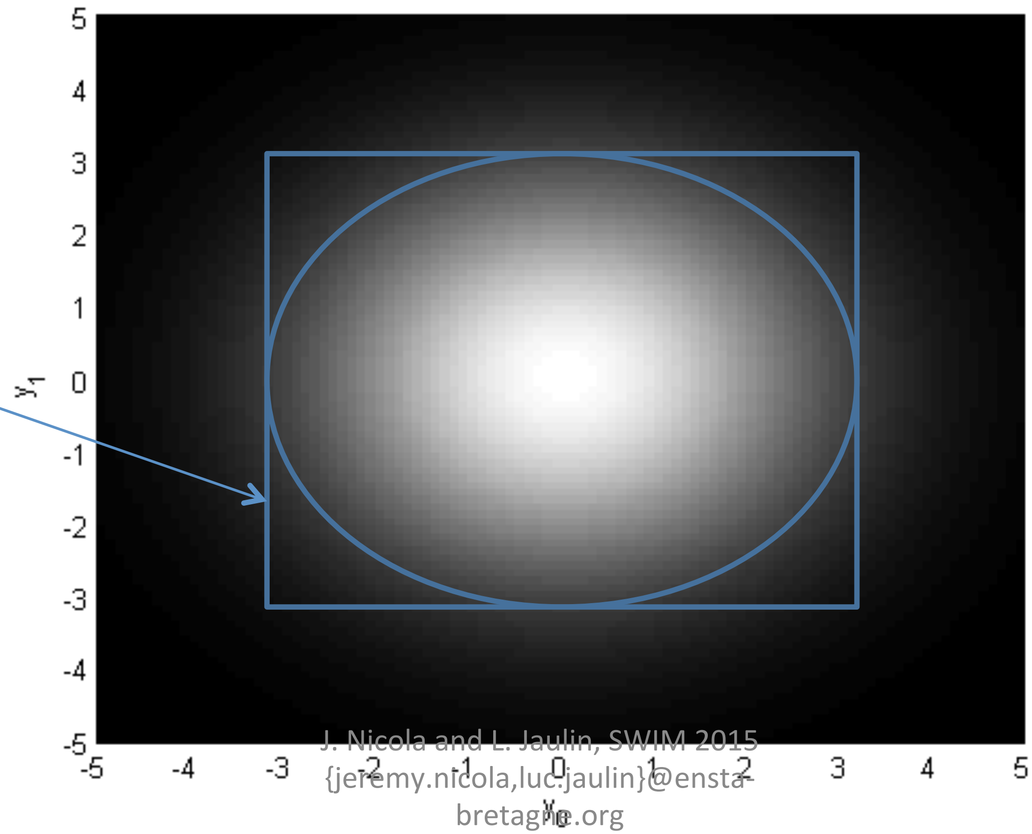
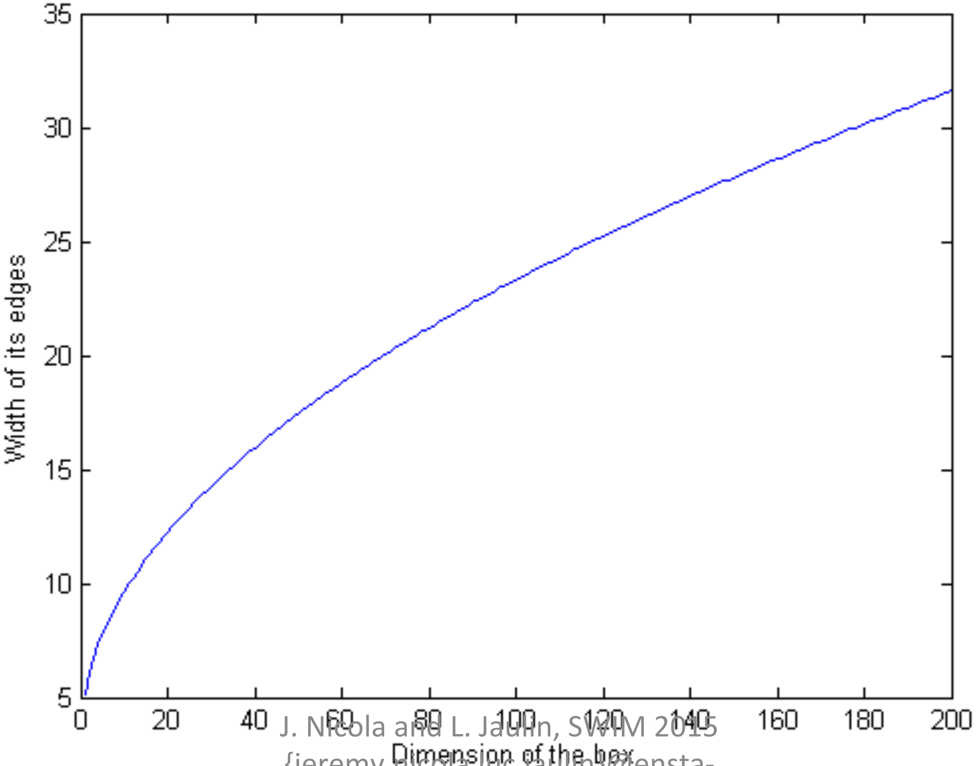


$$\boldsymbol{y} \in \tilde{\boldsymbol{y}} + 3.04 \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$$

• As the dimension (number of measurements) grows, so do the edges of the box



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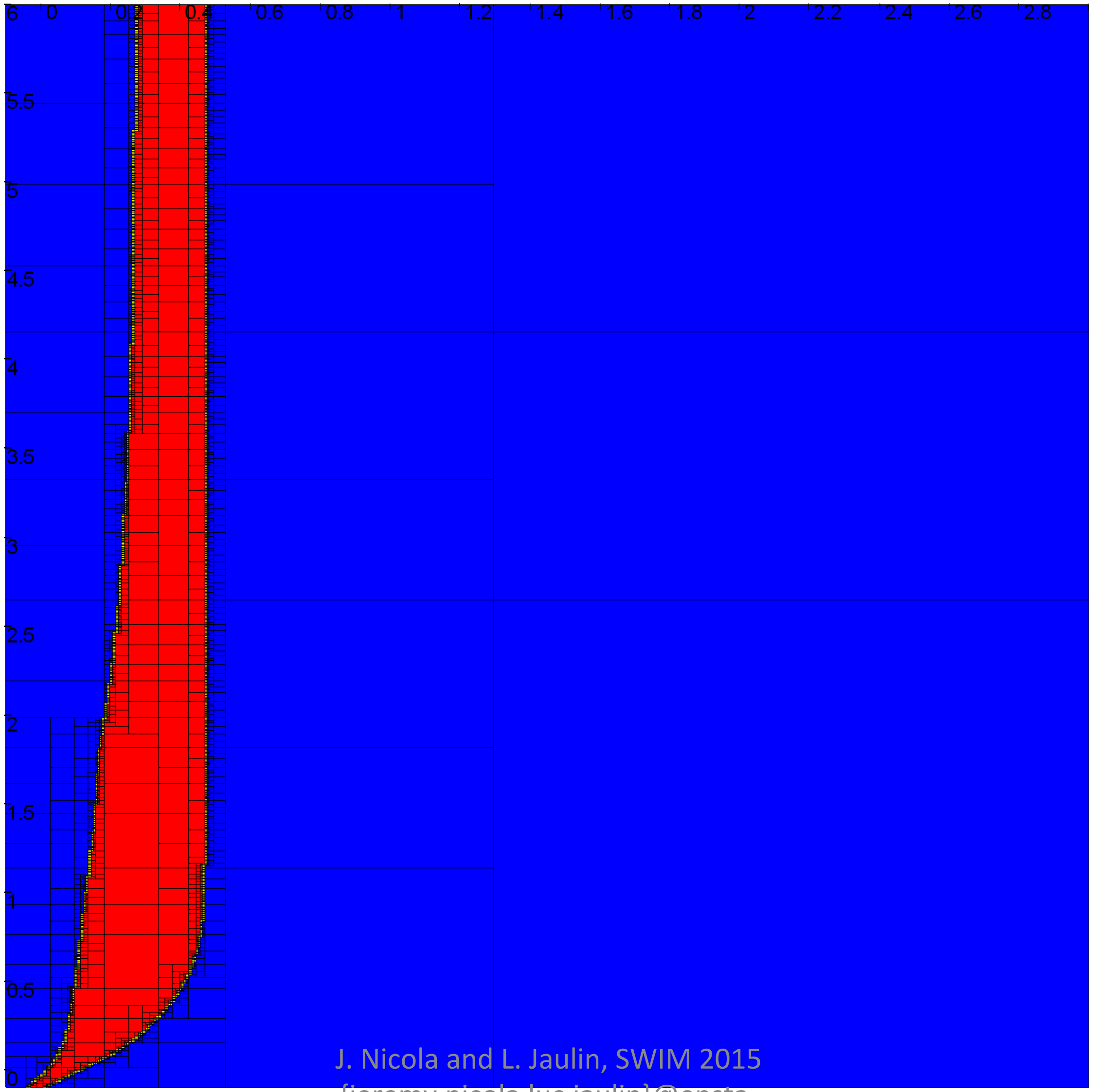
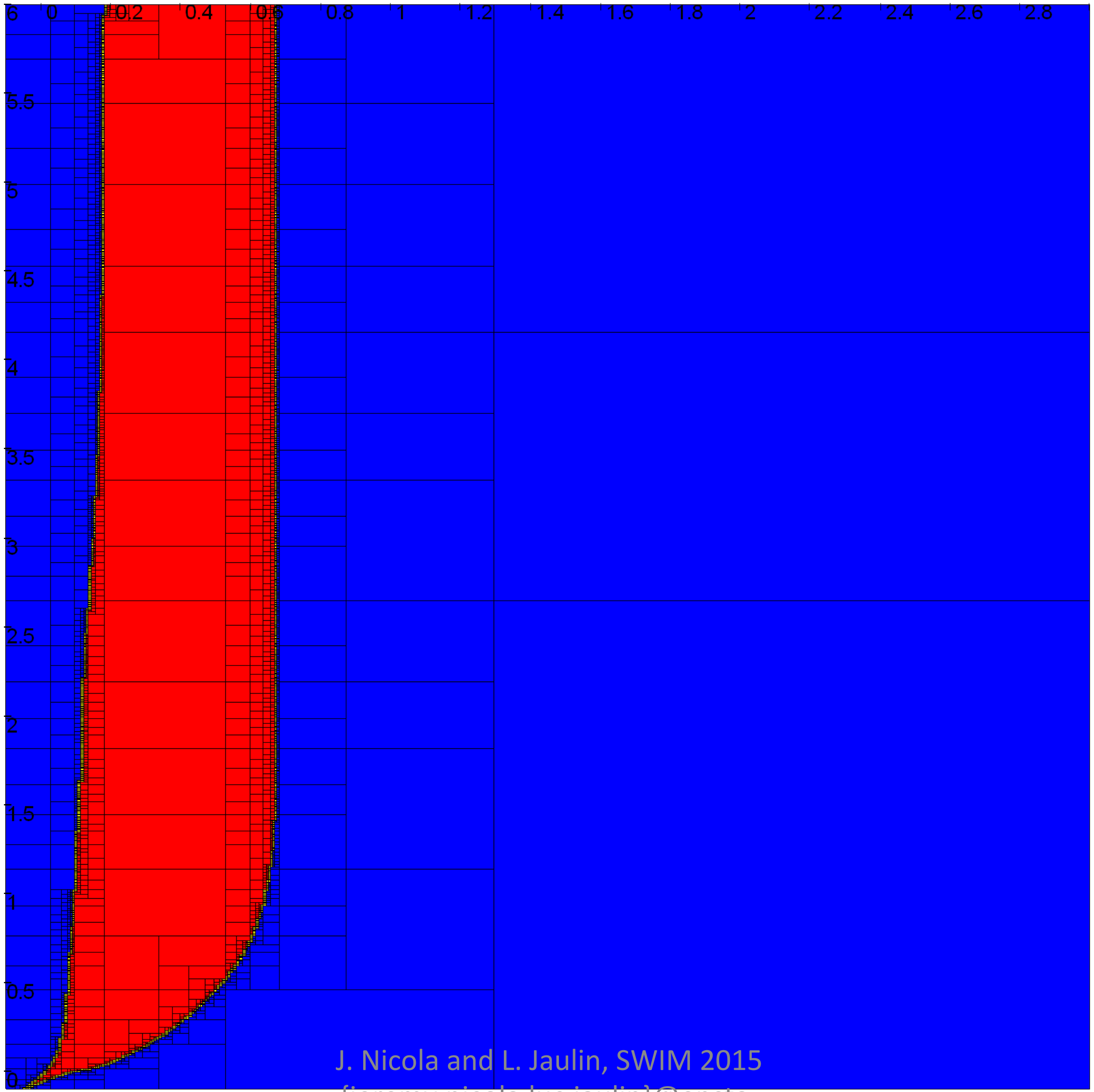
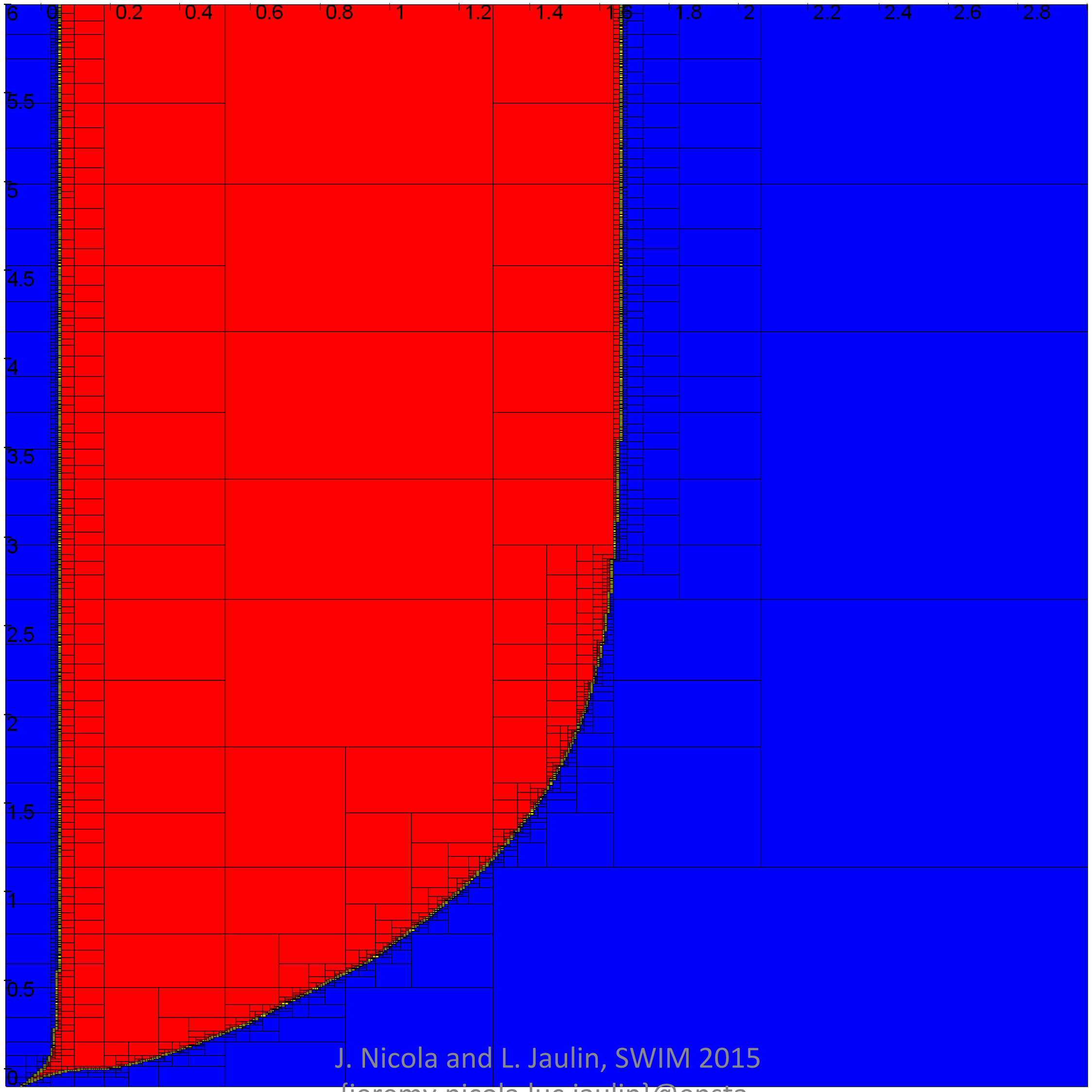
- We invert $y=g(x)$ with a 99% confidence using SIVIA



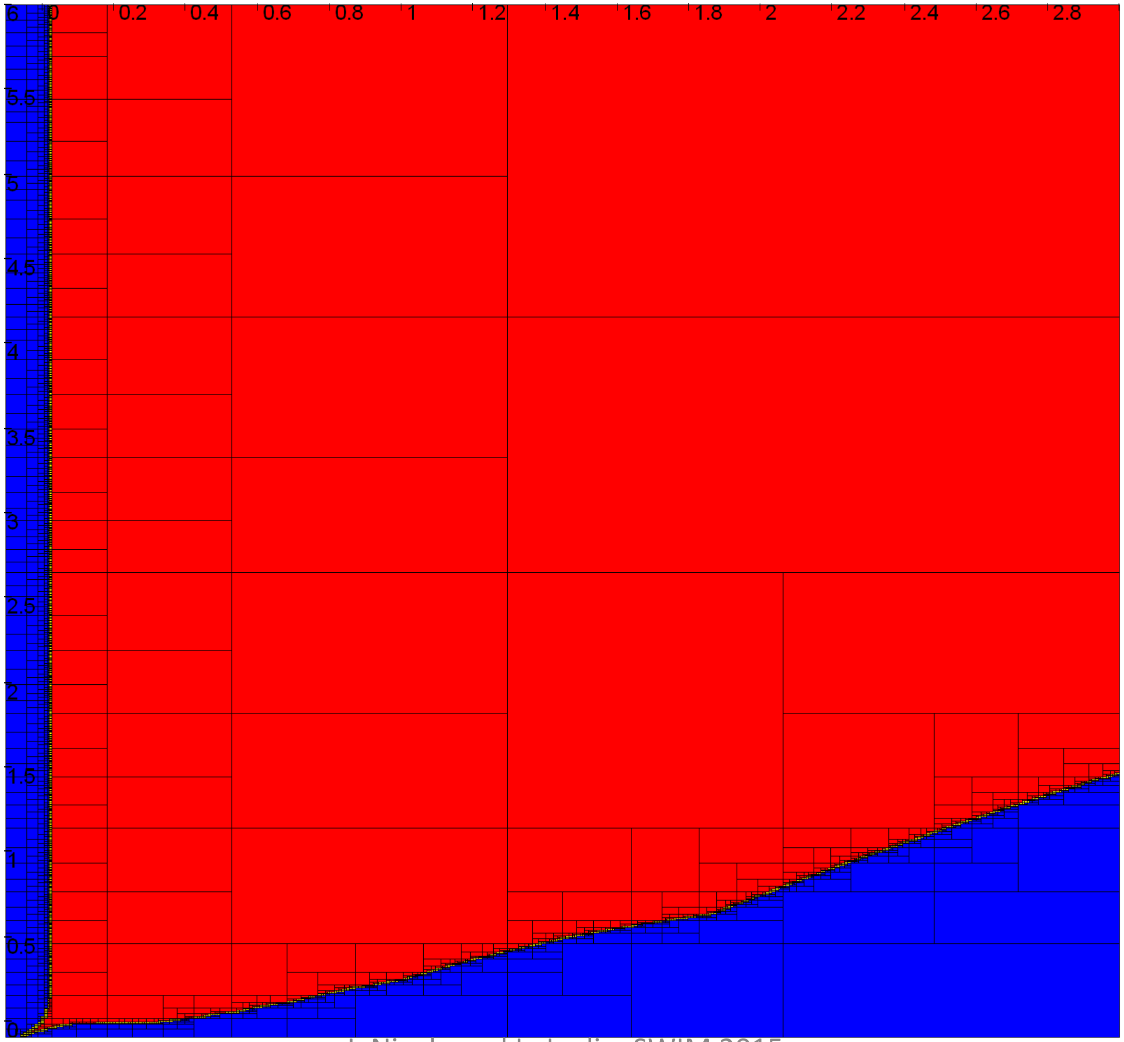
With 10 measurements



With 20 measurements



With 50 measurements



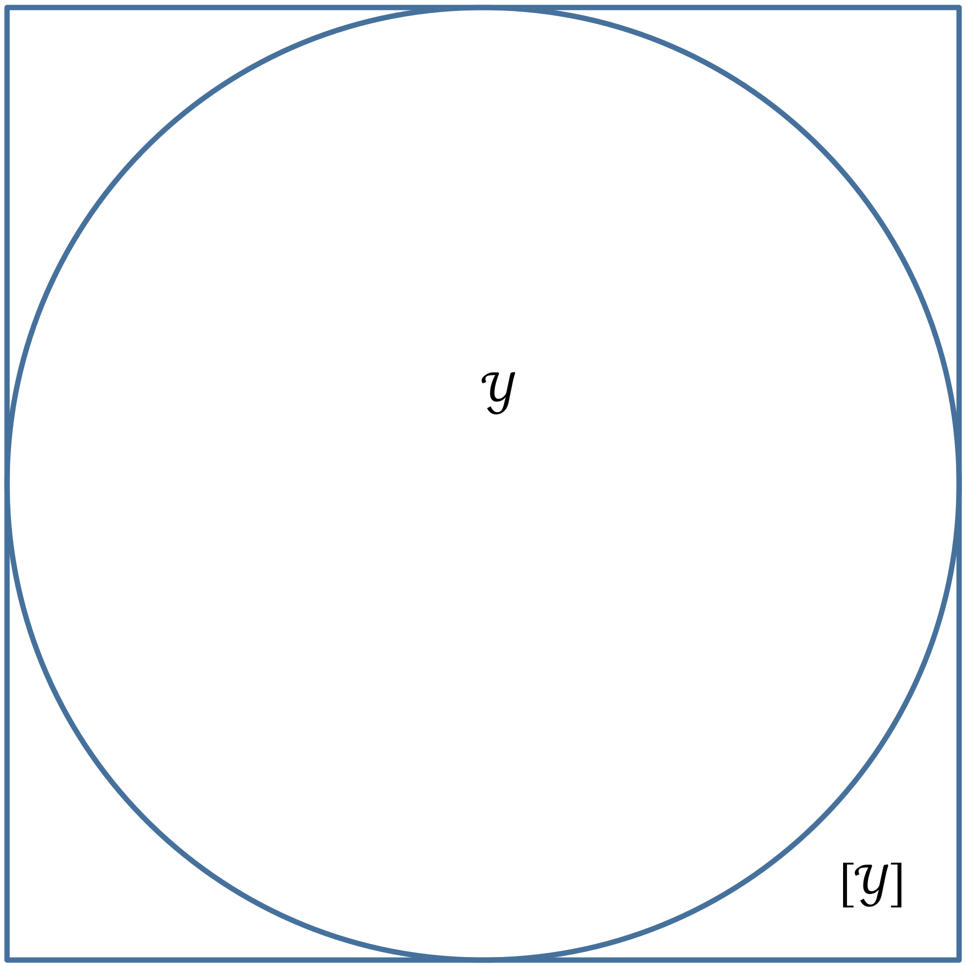
With 100 measurements

- The precision of the inversion does not grow with the number of measurements

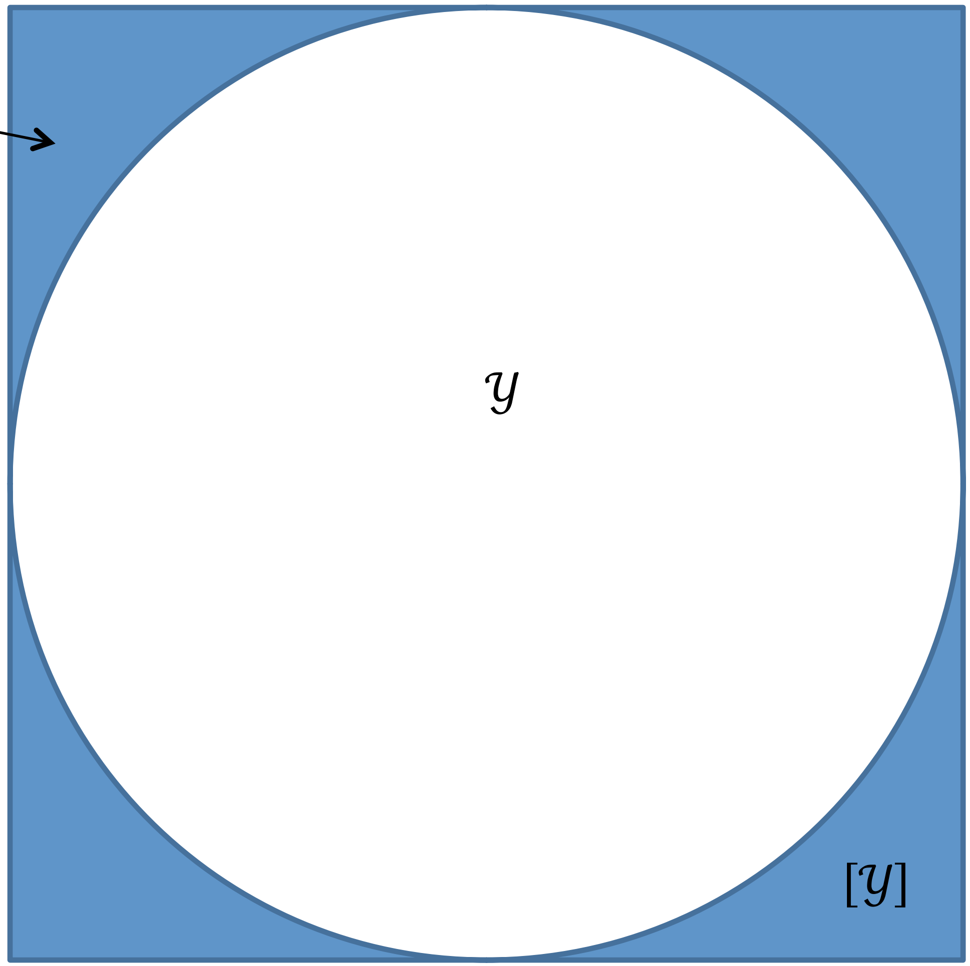
- The volume of the box-hull $[\mathcal{Y}]$ of $\mathcal{Y}$ grows as:

$$vol([\mathcal{Y}]) = (2 * a(\eta, k))^k$$

- A box is a poor approximation of a disk


$\mathcal{Y}$
$[\mathcal{Y}]$

The volume of
$[\mathcal{Y}]\backslash\mathcal{Y}$
is not neglictible
in high dimension



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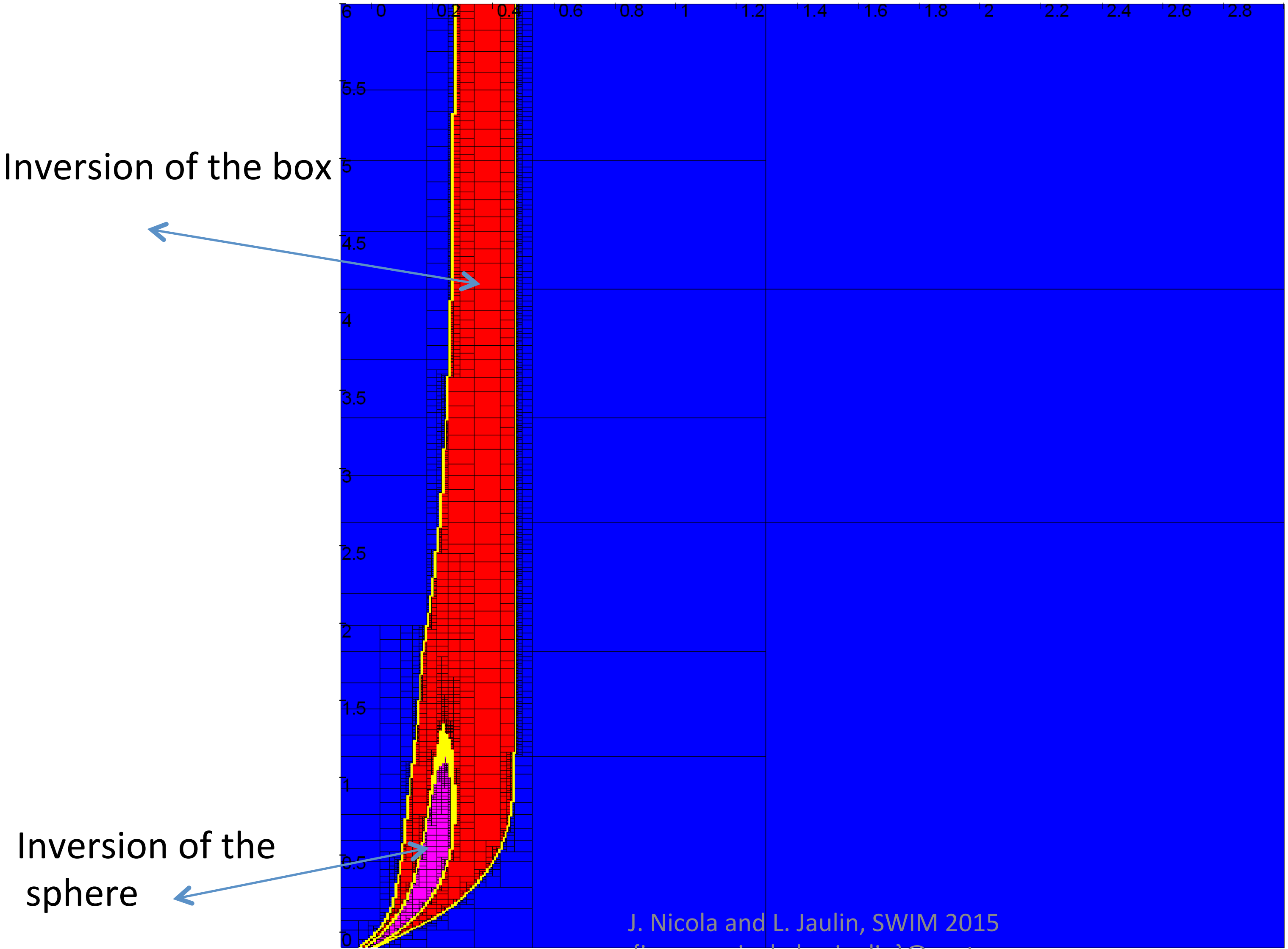
$$\Leftrightarrow$$

$$\begin{cases} y = g(x) \\ \sum (y_i - \tilde{y}_i)^2 \quad \leq a^2 \end{cases}$$



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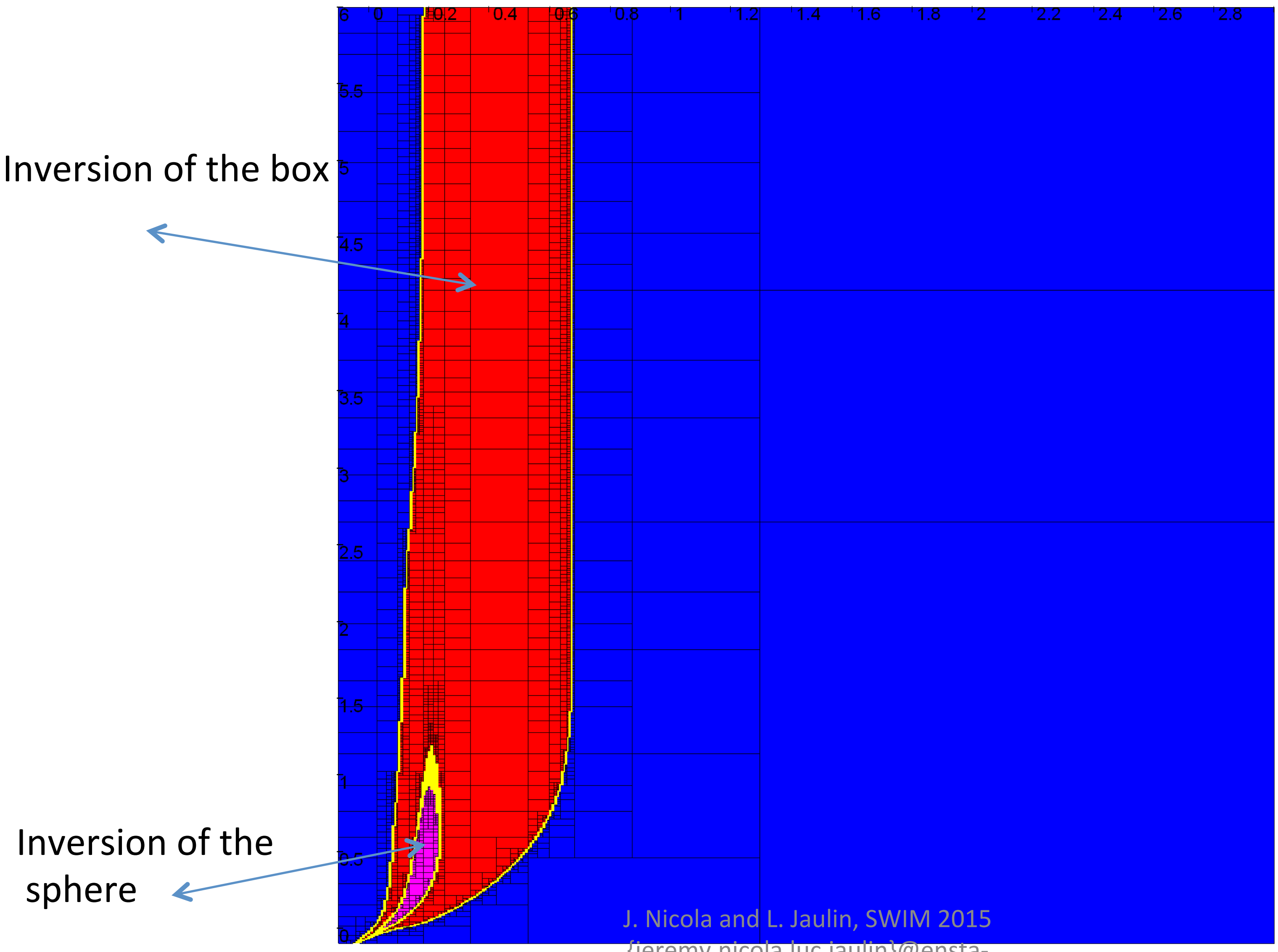
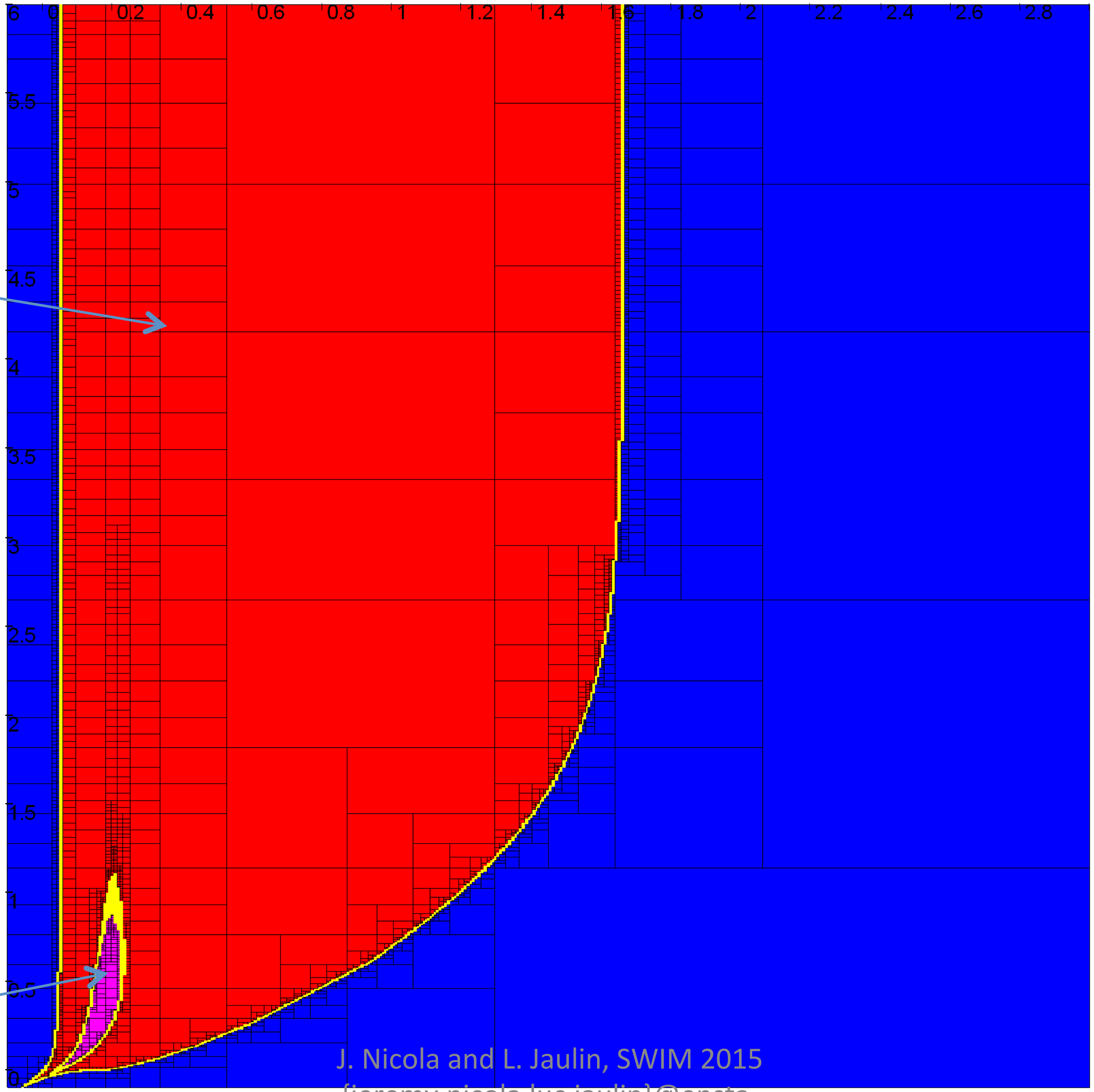


With 20 measurements

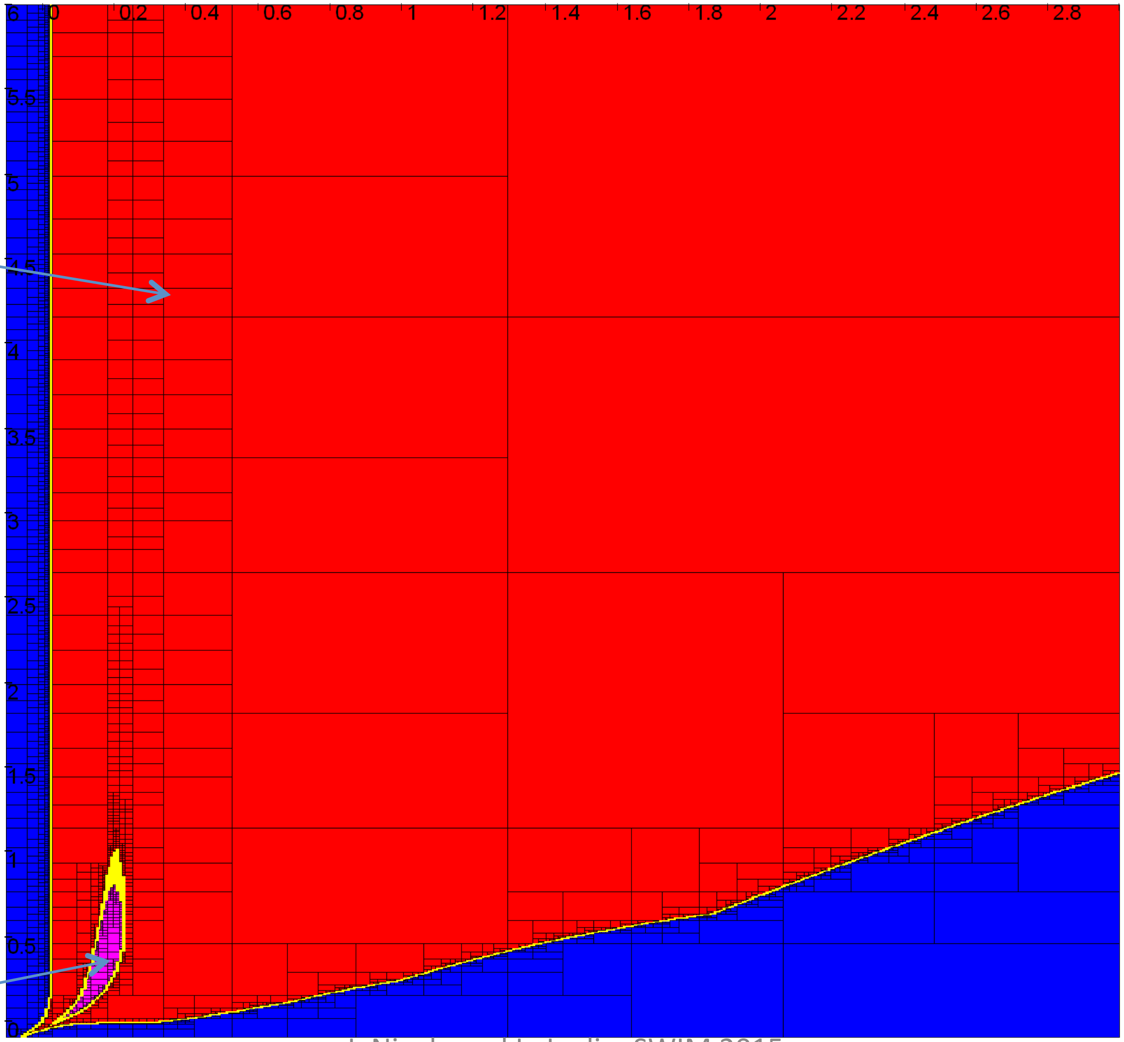
Inversion of the box



Inversion of the sphere

With 50 measurements

Inversion of the box



Inversion of the sphere

With 100 measurements

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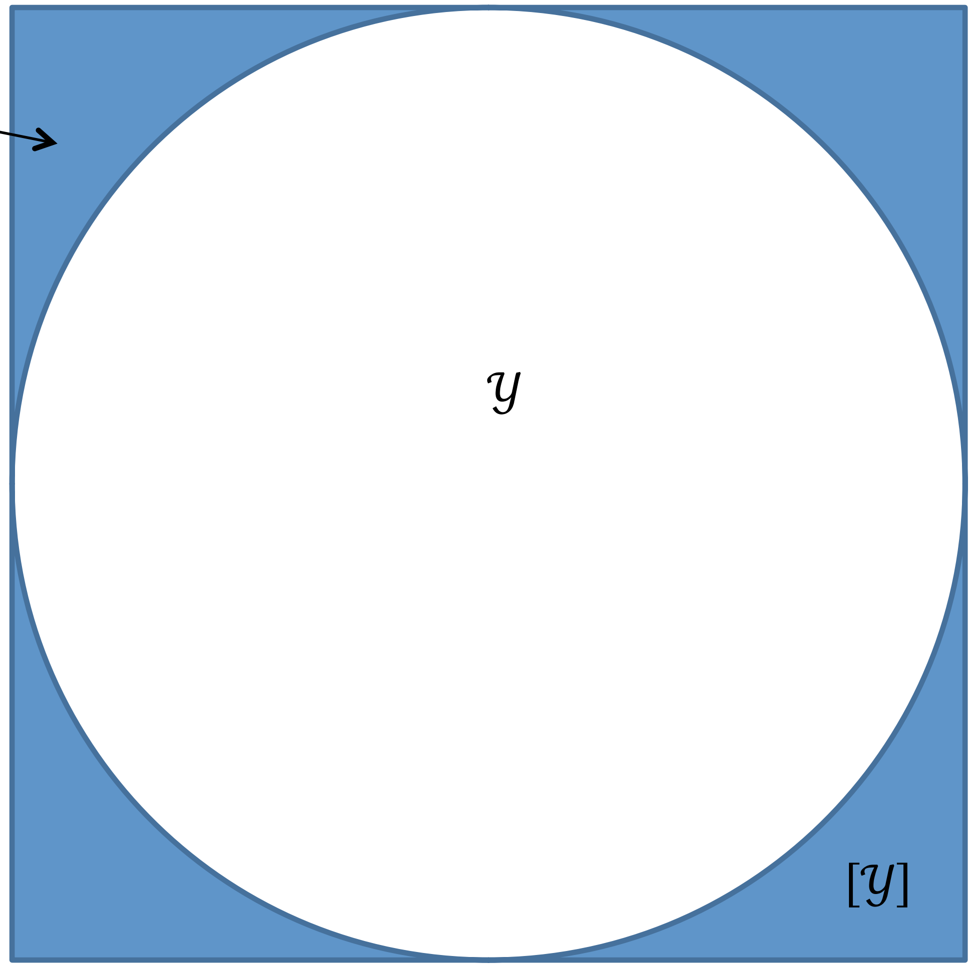
$$\lim_{n \to \infty} (V_n) = 0$$

- As the dimension of the sphere (number of measurements) grows, its volume tends towards zero

The volume of
$[\mathcal{Y}]\backslash\mathcal{Y}$
is not neglictible
in high dimension



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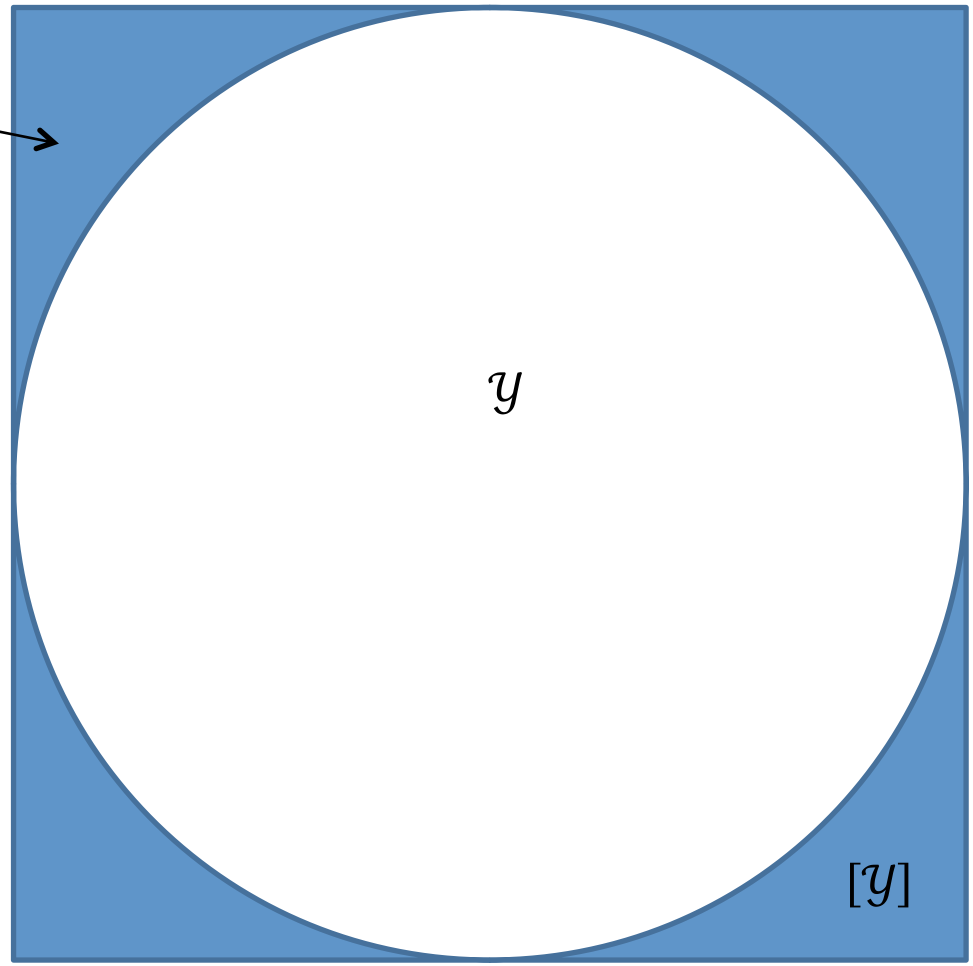

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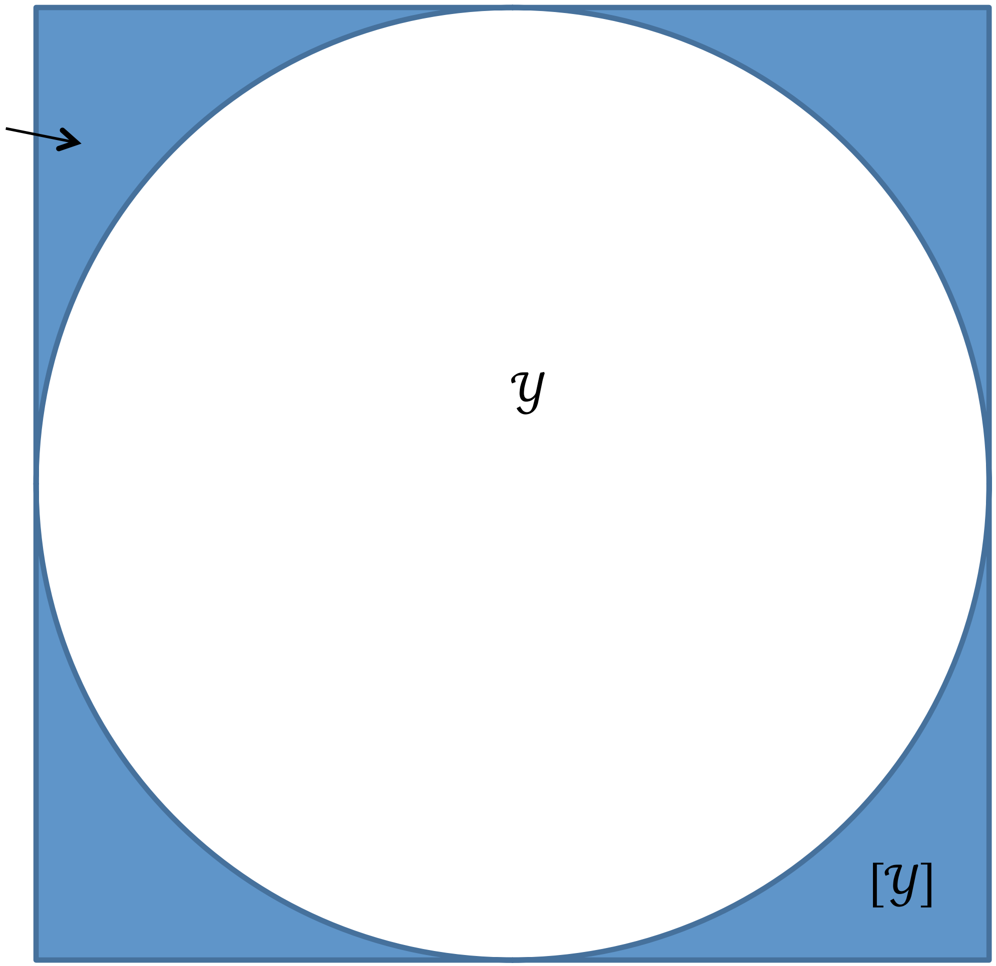
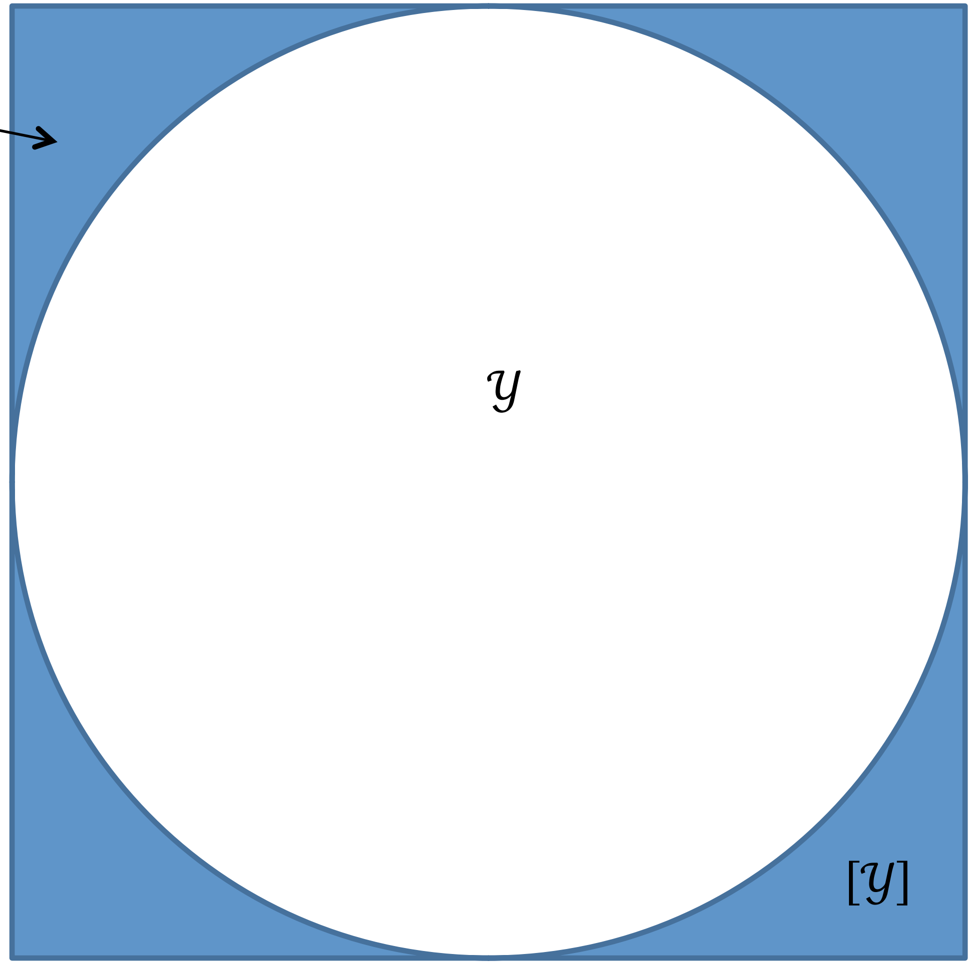


In high dimension

$vol([\mathcal{Y}]\backslash\mathcal{Y}) \to vol([\mathcal{Y}])$

~~The volume of $[\mathcal{Y}]\backslash\mathcal{Y}$ is not neglictible in high dimension~~



In high dimension

$vol([\mathcal{Y}]\backslash\mathcal{Y}) \rightarrow vol([\mathcal{Y}])$

A box is « filled by its corners »

- By adding a big number of measurements, we could inverse a virtually zero-volume set, i.e. reach an infinite precision

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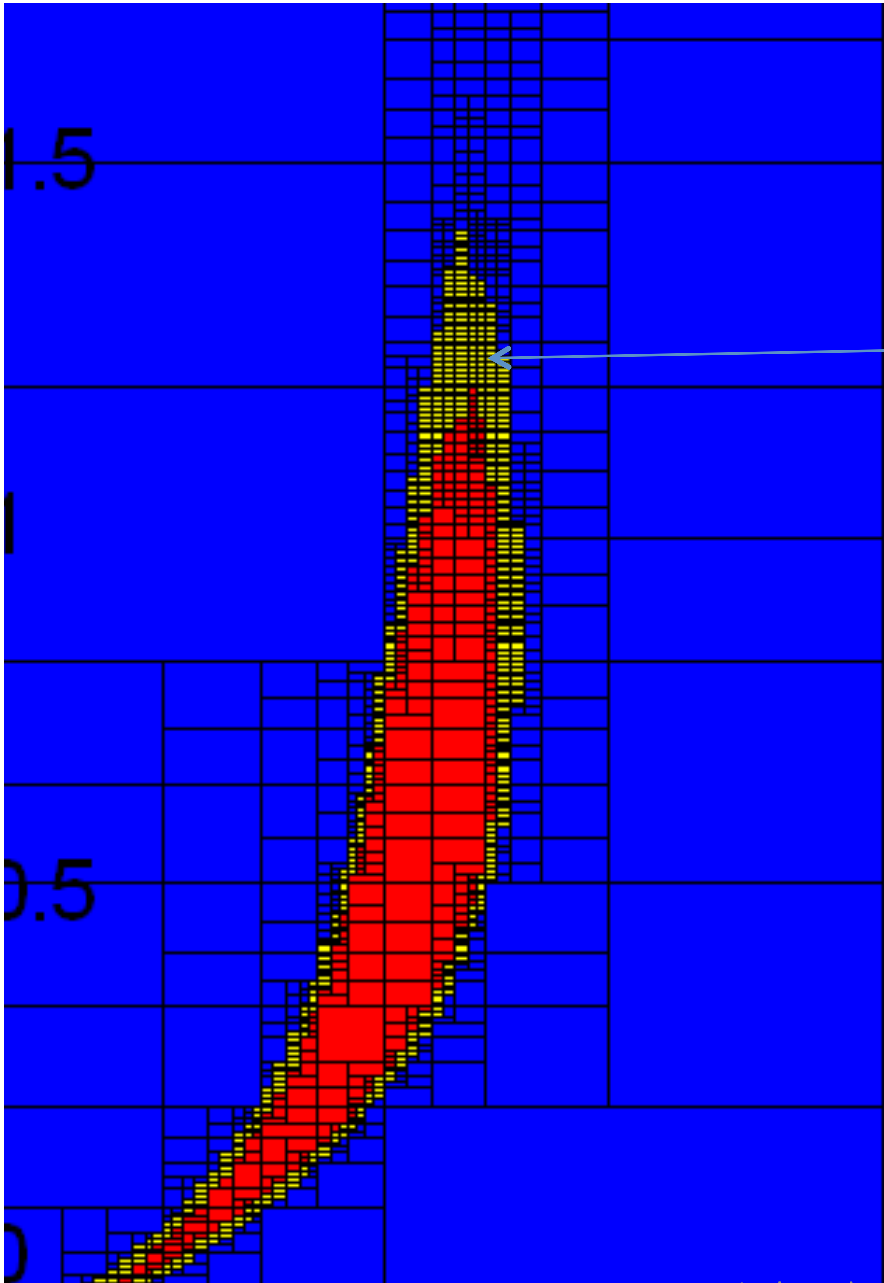
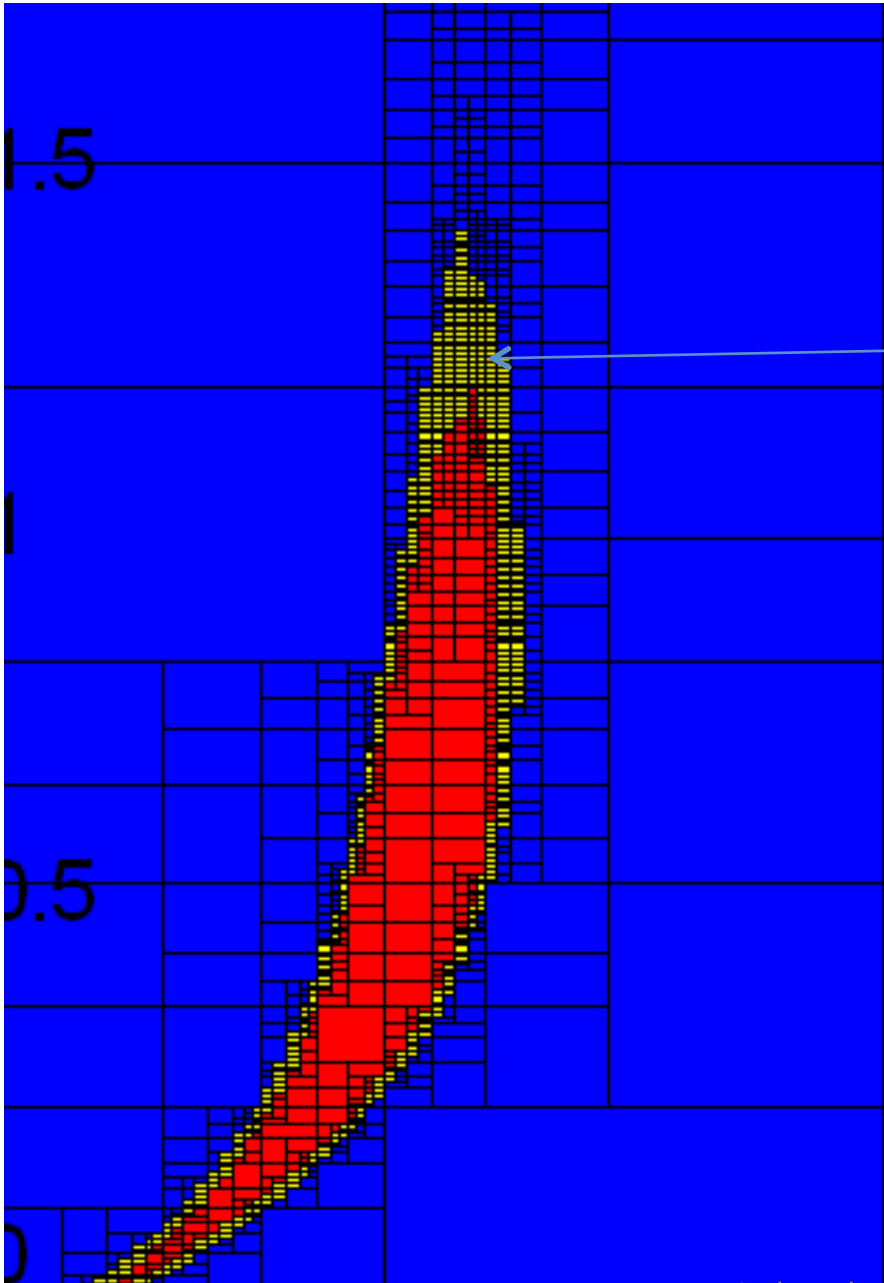
Multiple occurences

$$g\big(f(\boldsymbol{x})\big) = (f_1(x) - \tilde{y}_1)^2 + (f_2(x) - \tilde{y}_2)^2 + \cdots$$

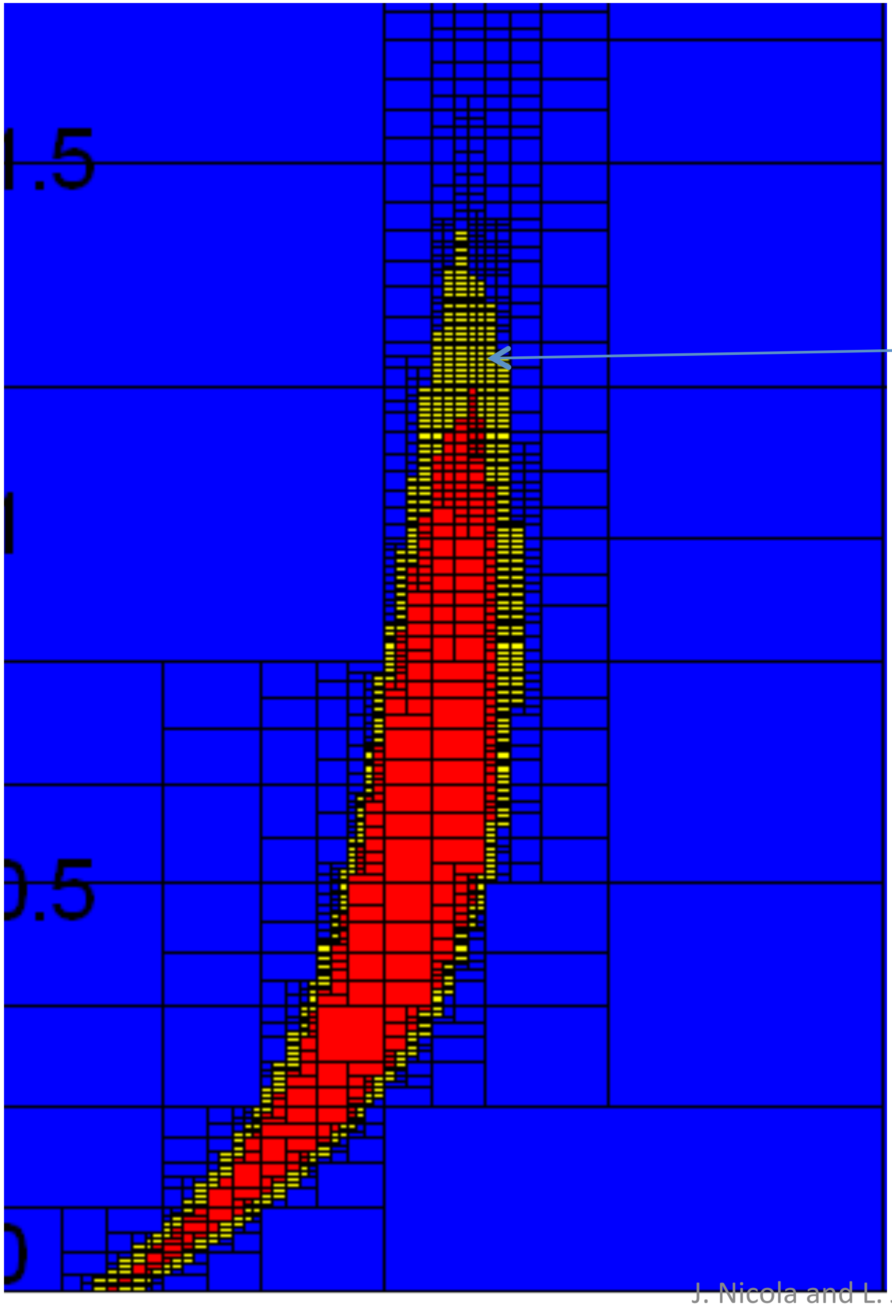


Wrapping effect



Wrapping effect

Wrapping effect is caused by the fact that a box
Is generaly a poor representation of a set



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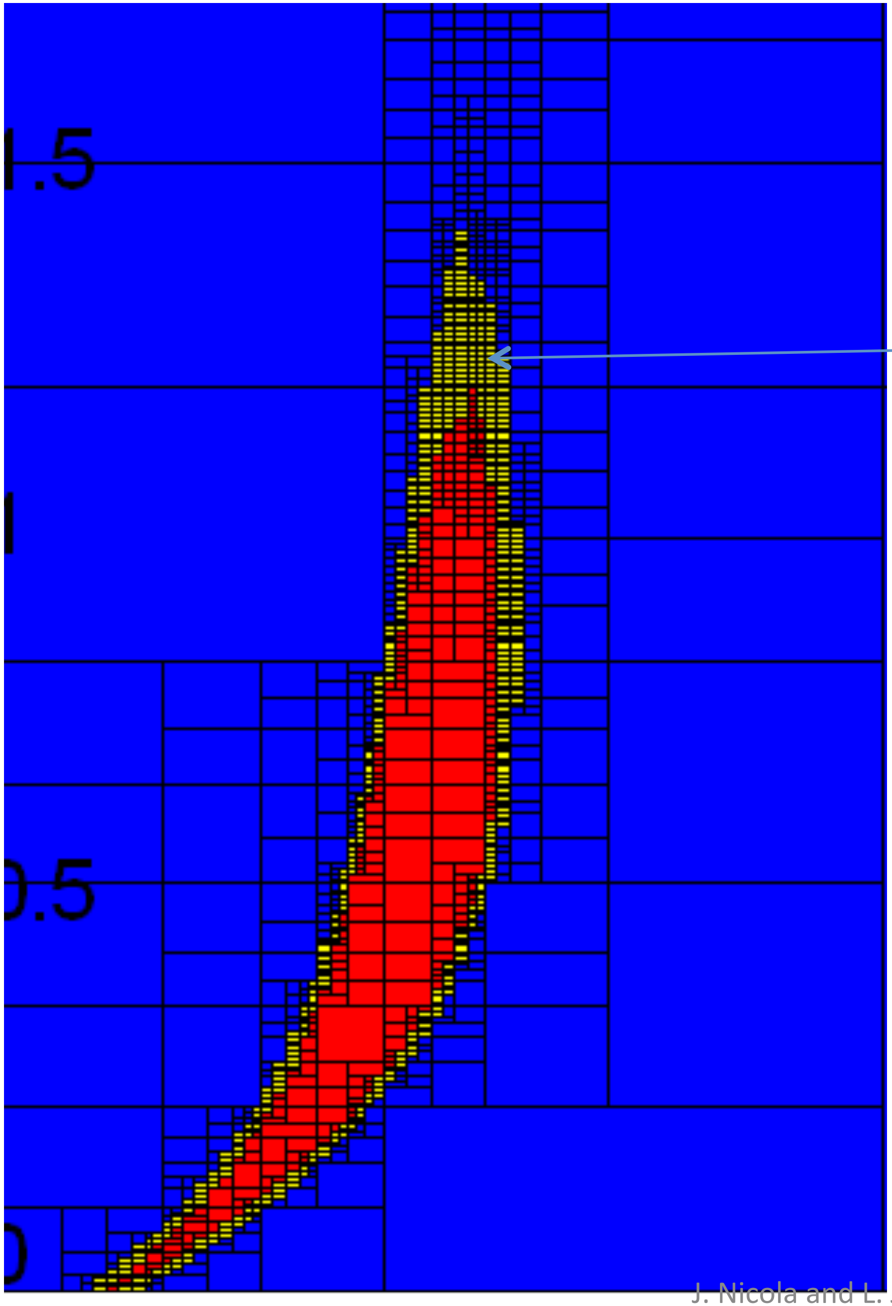
We could use linear methods (polytopes, ellipsoids, affine forms…) to reduce the wrapping effect



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An approach is to use the centred form of the function to invert:

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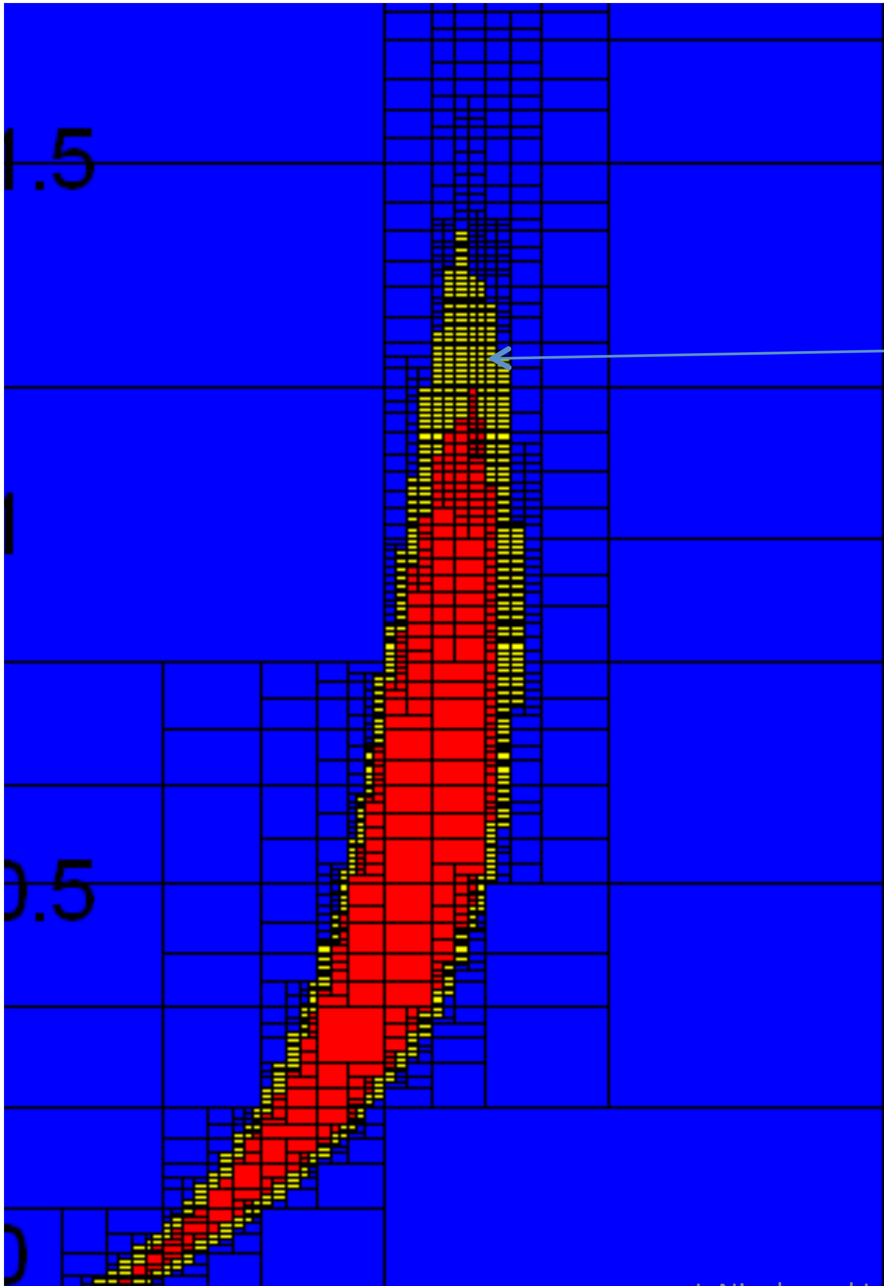

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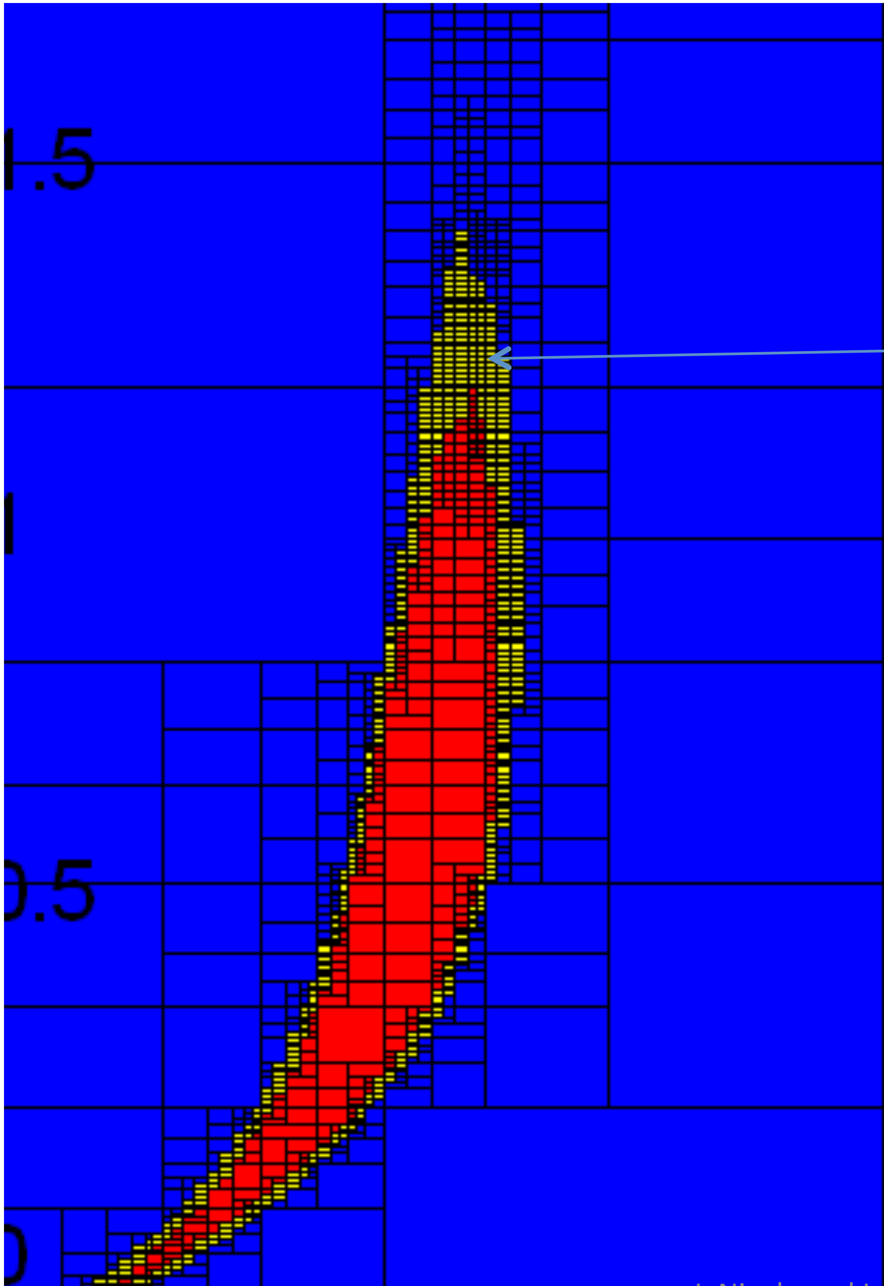

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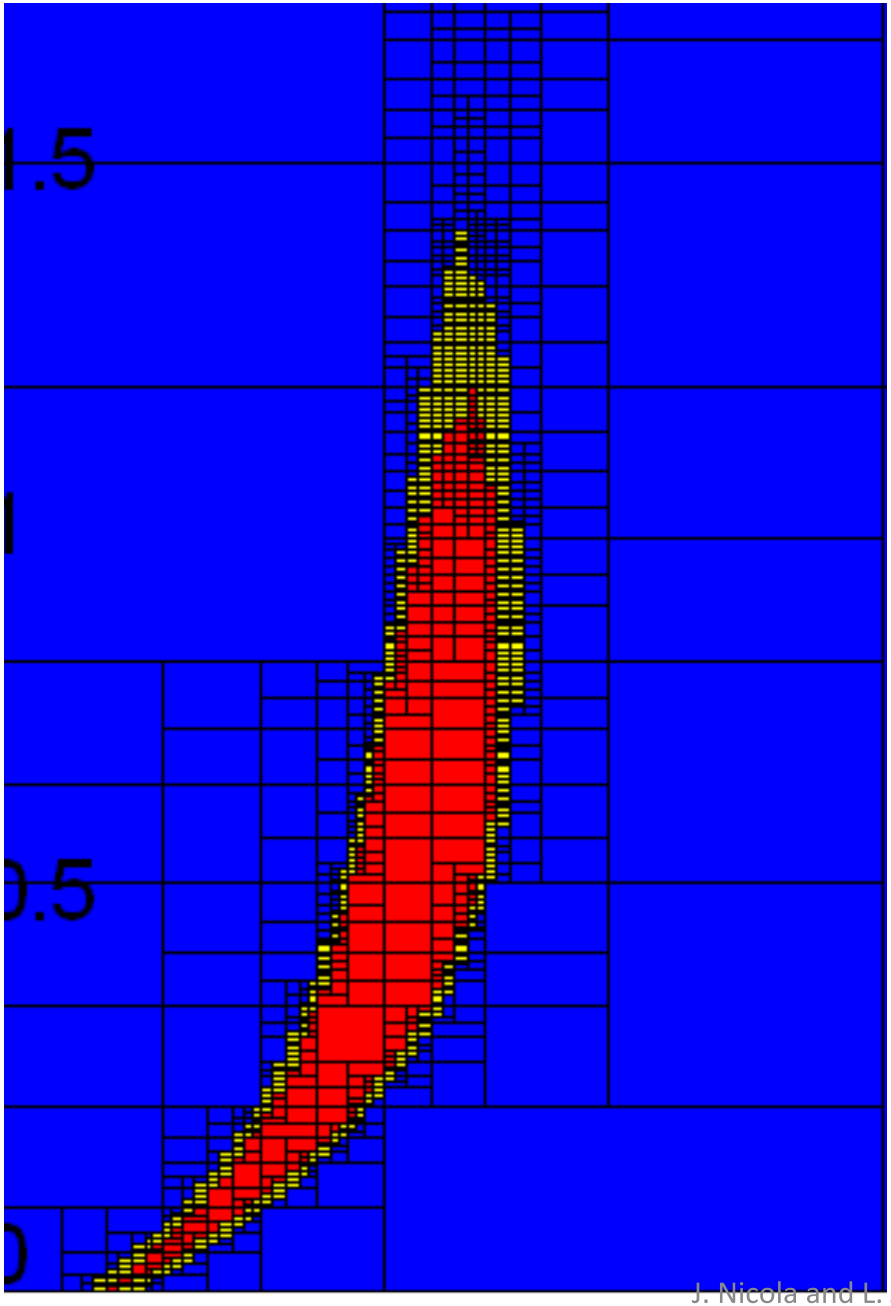
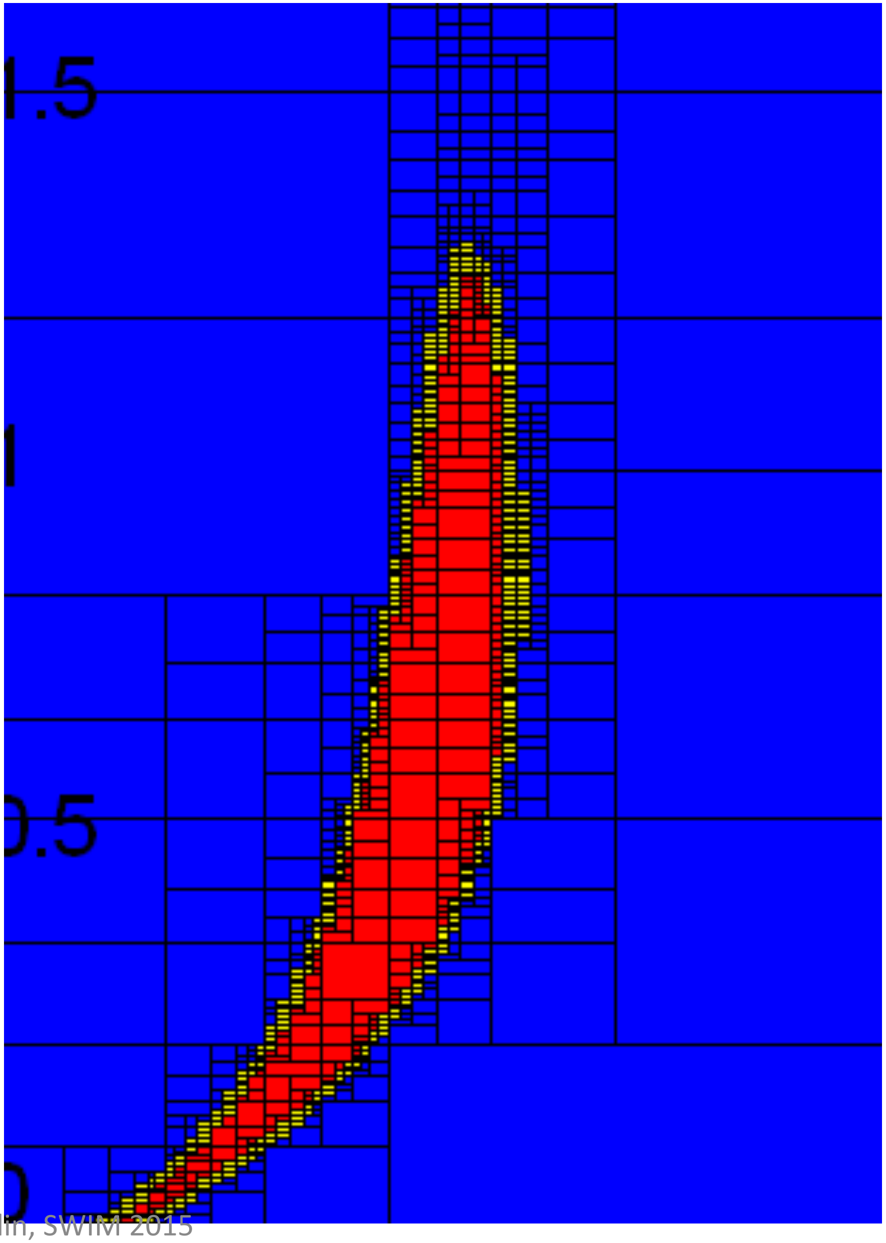
With **m**=mid([**x**]), **g** is the Jacobian of f

$$lim_{w([x])\to 0} \frac{w([f_c]([x]))}{w(f([x]))} = 1$$

$w([x])$ is the width of $[x]$



Same inversion with the centred-form



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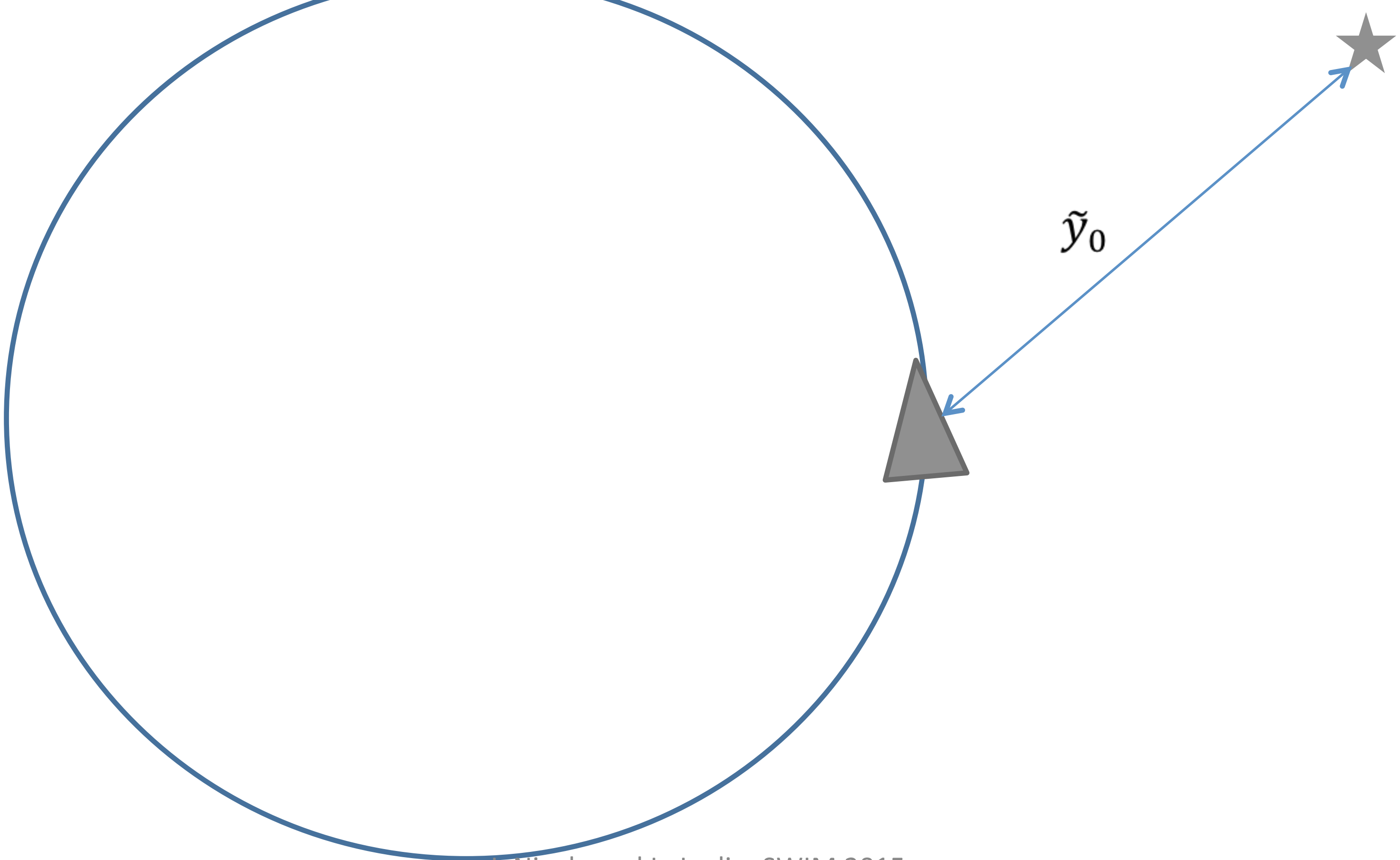
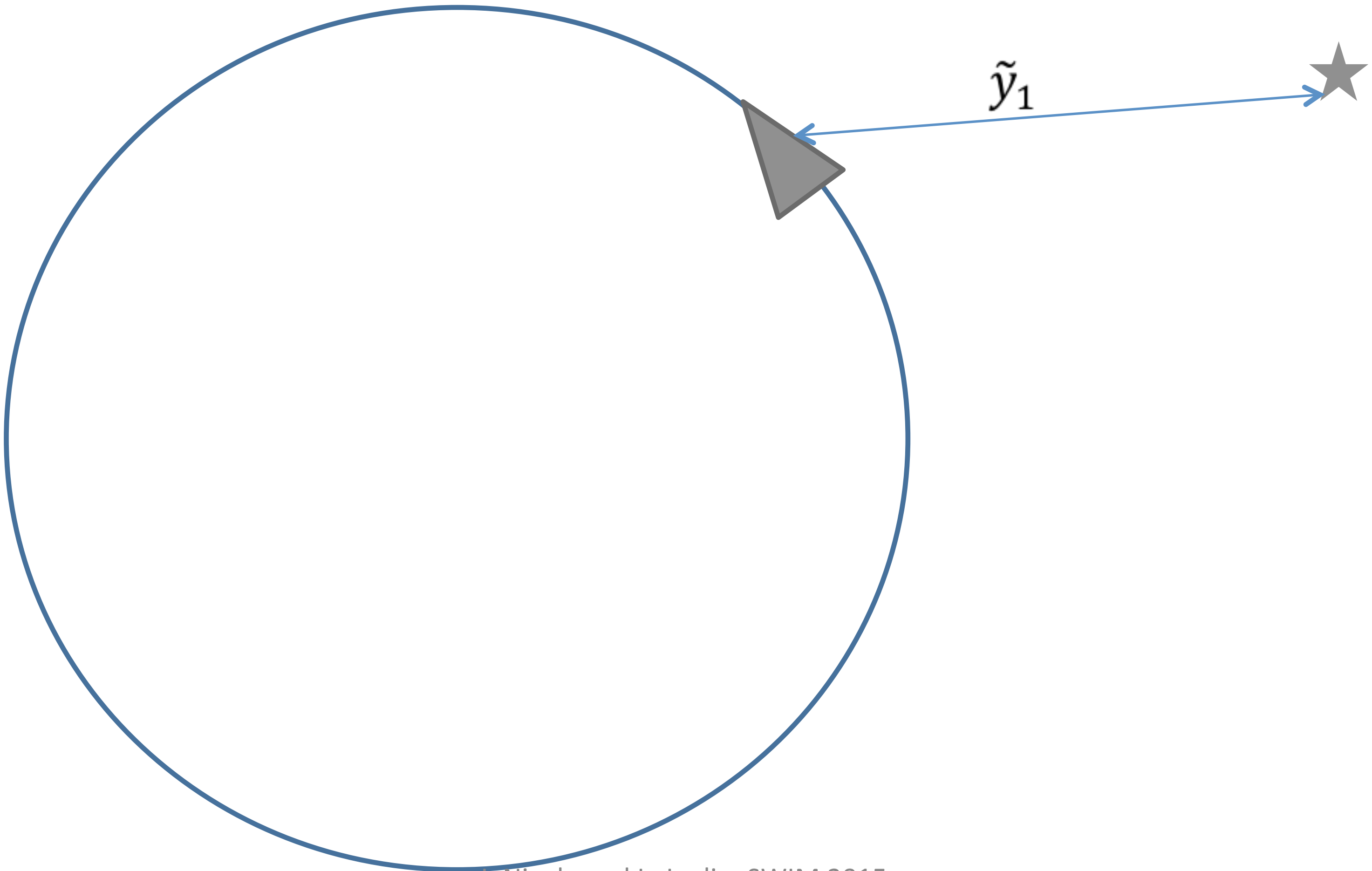
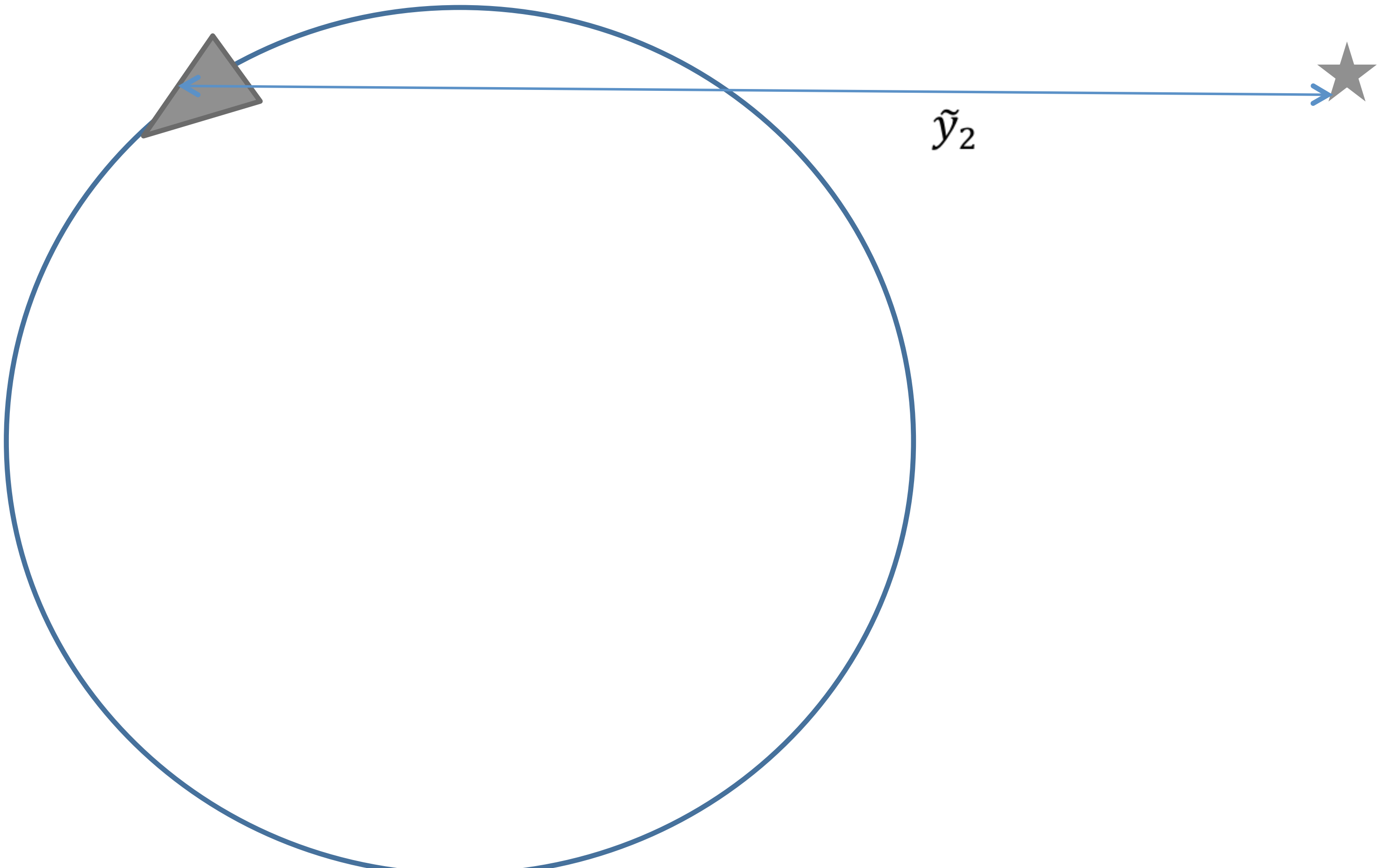
- A mobile robot moving in circles measures:
  - Its position with a GPS with a high precision
  - Its distance to a beacon, subject to a white normally distributed noise of variance 1

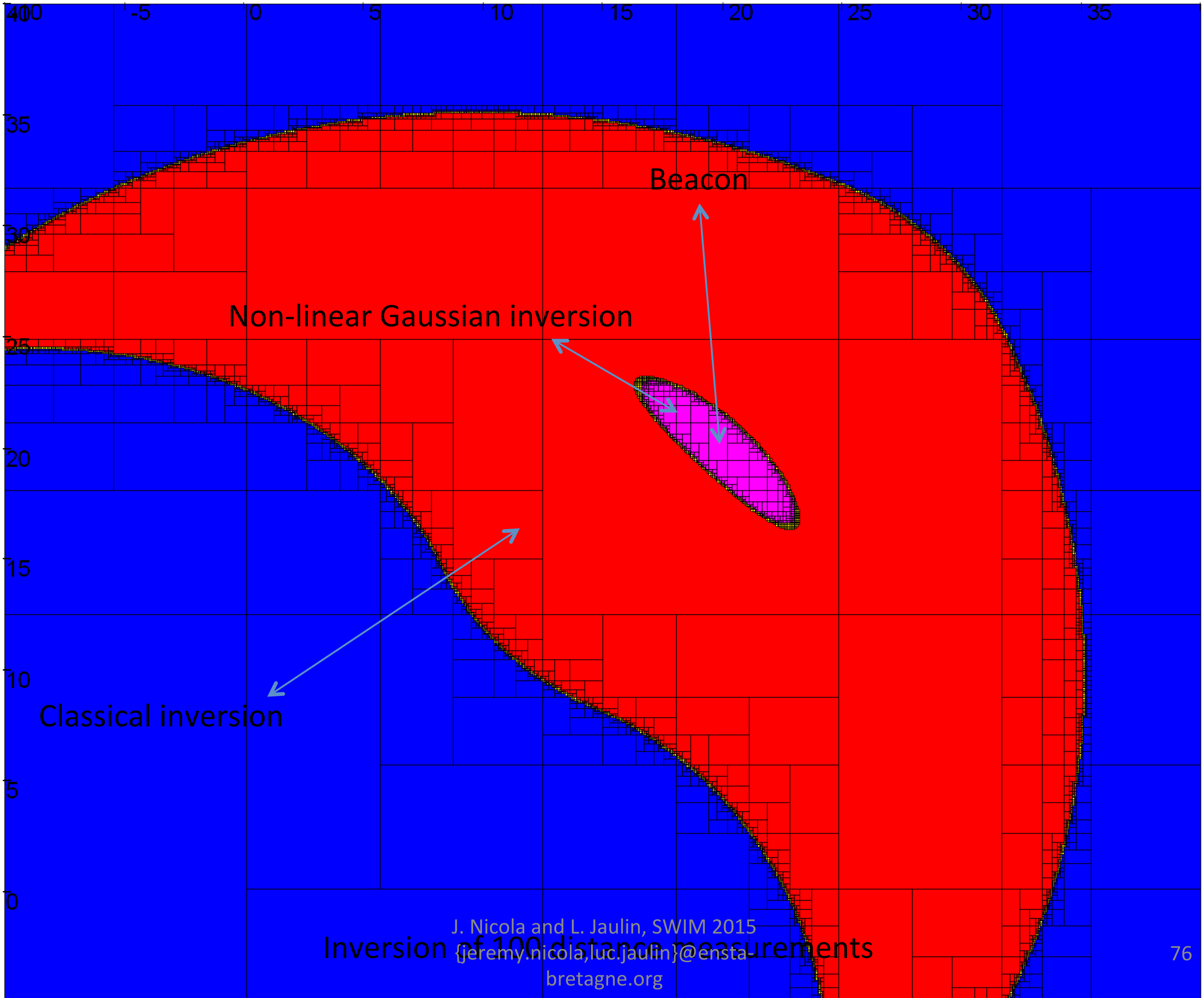

$\tilde{y}_0$


$\tilde{y}_1$



$\tilde{y}_2$

- The position of the beacon is initially unknown



Inversion of 100 distance measurements

- Context
- Problem
- Classical method
- Proposed method
- Improvements
- Application
- Conclusion

- We took a probabilistic property:
  - The noise is normally distributed, additive, white
- We casted this property as a geometrical constraint
- We are able to reliably and precisely invert a non-linear function in a least-square fashion, but without linearizing

Thank you for your attention