Sub-Interval Perturbation Method for Standard Eigenvalue Problem

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Abstract

- In vibration analysis, Finite Element Method (FEM) formulation of structures under dynamic states leads to generalized eigenvalue problem.

- Generally we have crisp values of material properties for structural dynamic problems.

- As a result of errors in measurements, observations, calculations or due to maintenance induced errors etc. we may have uncertain bounds.

- This paper deals with sub-interval perturbation procedure for computing upper and lower eigenvalue and eigenvector bounds.

Some Literature Review

Few literatures for solving structural dynamics problem based on interval analysis in perturbation approach of structural dynamics are available.

- Alefeld and Herzberger (1983) and Moore et al. (2009) presented a detailed discussion on interval computations.
- Qiu et al. (1996) proposed an interval perturbation approximating formula for evaluating interval eigenvalues for structures.

- An inner approximation algorithm has been proposed by Hladik et al. (2011) with perturbations belonging to some given interval.

- For structures with large interval parameters Qiu and Elishakoff (1998) proposed a subinterval perturbation for estimating static displacement bound.

Interval

A subset of *R* such that $A^I = [\underline{a}, \overline{a}] = \{ t \mid \underline{a} \le t \le \overline{a}, \underline{a}, \overline{a} \in R \}$ is called an interval.

Arithmetic operations on intervals: $A^I = [\underline{a}, \overline{a}]$, $B^I = [\underline{b}, \overline{b}]$

- $A^I + B^I = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$
- $A^I - B^I = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$
- $A^I \cdot B^I = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\},\ \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}]$
- $A^I / B^I = [\underline{a}, \overline{a}] \cdot [1/\overline{b}, 1/\underline{b}]$

Interval center:

$$A^c = \frac{\overline{a} + \underline{a}}{2}$$

Interval width :

$$A^w = \overline{a} - \underline{a}$$

Interval radius:

$$\Delta A = \frac{\overline{a} - \underline{a}}{2}$$

An interval $A^I = [\underline{a}, \overline{a}]$ may be represented in term of center and radius as

.

$$A^I = [A^c - \Delta A, A^c + \Delta A]$$

Standard interval eigenvalue problem :

$$K^I x_i^I = \lambda_i^I x_i^I, i = 1, 2, \ldots, n$$

where λ_i^I is the eigenvalue and x_i^I is the corresponding eigenvector.

Generalised interval eigenvalue problem :

$$K^I x_i^I = \lambda_i^I M^I x_i^I, i = 1, 2, \ldots, n$$

where λ_i^I is the eigenvalue and x_i^I is the corresponding eigenvector.

Perturbation

- An eigenvalue perturbation is the process of computing eigenvalue and its corresponding eigenvectors from some known eigenvalue and eigenvectors with small perturbation.

- Eigenvalues λ_i^c and eigenvectors x_i^c obtained from crisp center matrix K^c are considered as unperturbed eigenvalues.

- Perturbations are done with respect to crisp values to obtain lower and upper bounds of eigenvalues and vectors of standard eigenvalue problem with interval parameters.

Interval Perturbation Procedure

Let us consider a standard interval eigenvalue problem

$$K^I x_i^I = \lambda_i^I x_i^I, \quad i = 1, 2, \ldots, n \tag{1}$$

In term of interval center and radius, equation (1) may be written as

$$(K^c + \delta K)(x_i^c + \delta x_i) = (\lambda_i^c + \delta\lambda_i)(x_i^c + \delta x_i) \tag{2}$$

satisfying $(x_i^c)^T K^c x_j^c = \lambda_i^c \delta_{ij}$ where $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ is the Kronecker Delta function.

$$\delta K = [-\Delta K, \Delta K] \qquad \delta \lambda_i = [-\Delta \lambda_i, \Delta \lambda_i]$$

- Using and where

$$\Delta \lambda_i = (x_i^c)^T \Delta K \; x_i^c$$

.

- The required first order perturbation of eigenvalues may be given by

(3a)
$$\underline{\lambda}_i = \lambda_i^c - (x_i^c)^T \Delta K \; x_i^c$$

(3b)
$$\overline{\lambda}_i = \lambda_i^c + (x_i^c)^T \Delta K \; x_i^c$$

- The required first order perturbation of eigenvectors may be given by

$$\underline{x}_i = x_i^c + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c \qquad (4a)$$

$$\bar{x}_i = x_i^c - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c \qquad (4b)$$

- The first order upper and lower bounds

$$\underline{x}_i = \min\left\{ x_i^c - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c,\ x_i^c + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c \right\} \quad (5a)$$

$$\overline{x}_i = \max\left\{ x_i^c - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c,\ x_i^c + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x_j^c)^T \Delta K\, x_i^c}{\lambda_i^c - \lambda_j^c} x_j^c \right\} \quad (5b)$$

Sub-interval Perturbation

- Let $A^I = [\underline{a}, \overline{a}]$ be an interval, then its subintervals may be obtained by dividing the interval into m equal parts with width $(\overline{a} - \underline{a}) / m$.

- For an interval matrix $K^I = [\underline{K}, \overline{K}]$ of order n, the subinterval matrices may be obtained as

$$K^I = [\underline{K}, \overline{K}] = \bigotimes_{t=1}^{m} K_t^I$$

where $K_t^I = [\underline{K} + (t-1)(\overline{K} - \underline{K}) / m, \underline{K} + t(\overline{K} - \underline{K}) / m]$

and subinterval iteration $t = 1, 2, \ldots, m.$

- The interval perturbation procedure is then implemented over each subinterval K_t^I.

Inner approximation for eigenvalues

- λ_i^I for global (without sub-intervals) interval matrix K^I.

Outer approximation for eigenvalues

- $\lambda_i^I = [\min \underline{\lambda}_{it}, \max \overline{\lambda}_{it}]$, where $t = 1, 2, \ldots, m$ and m being sufficiently large.

Standard interval eigenvalue problem

- Consider a spring-mass system having four degrees of freedom as given in (Qiu et al. 1996) with mass matrix as crisp identity matrix and interval stiffness matrix K^I.

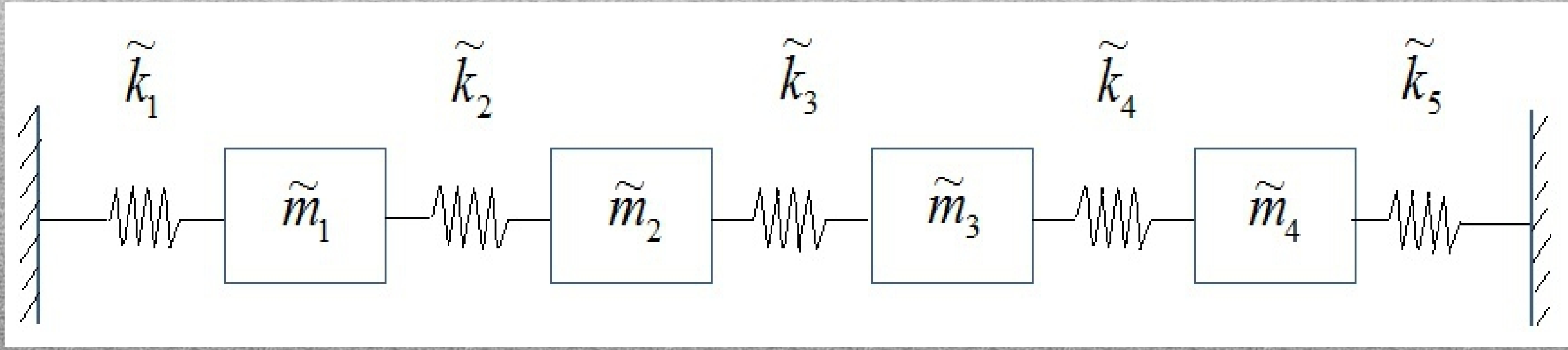

Figure 1. Four-degree spring-mass system

- In term of centre and radius, K^I and M^I may be written as $[M^c - \Delta M, M^c + \Delta M]$ and $[K^c - \Delta K, K^c + \Delta K]$ respectively.

- $$M^c = diag\,(1,1,1,1),\ \ \Delta M = diag\,(0,0,0,0)$$

$$K^c = \begin{bmatrix} 3000 & -2000 & 0 & 0 \\ -2000 & 5000 & -3000 & 0 \\ 0 & -3000 & 7000 & -4000 \\ 0 & 0 & -4000 & 9000 \end{bmatrix}$$ and $$\Delta K = \begin{bmatrix} 25 & 15 & 0 & 0 \\ 15 & 35 & 20 & 0 \\ 0 & 20 & 45 & 25 \\ 0 & 0 & 25 & 55 \end{bmatrix}$$

Table 1. Inner approximation of eigenvalue and eigenvector bound

	i	1	2	3	4
Eigenvalues	$\underline{\lambda}_i$	843.06917	3342.7724 7	7032.4785 1	12621.679 85
	$\overline{\lambda}_i$	967.25379	3436.9242 8	7096.4391 2	12659.382 81
Eigenvectors	$\underline{x}_i$	-0.60066	-0.72150	0.35119	-0.05650
		-0.62436	0.13165	-0.72092	0.26693
		-0.45850	0.55212	0.25465	-0.64818
		-0.22777	0.39391	0.53599	0.70957
	$\overline{x}_i$	-0.59002	-0.71617	0.35801	-0.05499
		-0.62278	0.14858	-0.72034	0.27053
		-0.44999	0.55673	0.26436	-0.64631
		-0.22116	0.39669	0.53659	0.71275

Table 2. Outer approximation of eigenvalue and eigenvector bounds

	i	1	2	3	4
Eigenvalues	$\underline{\lambda}_i$	842.92509	3342.81139	7032.5430 4	12621.720 47
	$\overline{\lambda}_i$	967.10824	3436.9637 6	7096.5043 0	12659.423 70
Eigenvectors	$\underline{x}_i$	-0.60068	-0.72145	0.35118	-0.05651
		-0.62432	0.13169	-0.72089	0.26693
		-0.45847	0.55210	0.25467	-0.64817
		-0.22776	0.39389	0.53599	0.70957
	$\overline{x}_i$	-0.59004	-0.71619	0.35800	-0.05499
		-0.62274	0.14862	-0.72032	0.27053
		-0.44996	0.55671	0.26438	-0.64630
		-0.22115	0.39667	0.53658	0.71275

Table 3. Comparison of perturbed interval eigenvalue bounds

	Bounds	Present Inner approximation	Present Outer approximation	Hladik et al. (2011) Inner approximation	Qiu et al. (1996)
Eigenvalues	$\underline{\lambda}_1$	843.0692	842.9251	842.9251	826.7372
	$\overline{\lambda}_1$	967.2538	967.1082	967.1082	983.5858
	$\underline{\lambda}_2$	3342.7725	3342.8114	3337.0785	3331.1620
	$\overline{\lambda}_2$	3436.9243	3436.9638	3443.3127	3448.5350
	$\underline{\lambda}_3$	7032.4785	7032.5430	7002.2828	7000.1950
	$\overline{\lambda}_3$	7096.4391	7096.5043	7126.8283	7128.7230
	$\underline{\lambda}_4$	12621.6799	12621.7205	12560.8377	12588.2900
	$\overline{\lambda}_4$	12659.3828	12659.4237	12720.2273	12692.7700

Conclusion

- This investigation presents sub-interval perturbation procedure for obtaining inner and outer approximation of eigenvalue bounds for standard interval eigenvalue problems.

- Corresponding perturbed eigenvectors are also be computed.

- The perturbation of subintervals may not give exact bounds as higher order perturbations are neglected but provides a tighter first order inner approximation interval bounds with a small perturbation with respect to known crisp unperturbed eigenvalues and vectors.

- The proposed procedure may also be applied to other practical eigenvalue problems involving interval material properties .

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Thank you