Intervals of Sign Regular Matrices

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Outline

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3. Classes of matrices possessing the interval property
4. A conjecture which dates back to 1982 and its solution
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Notations

\( \mathbb{IR} \): set of the compact, nonempty real intervals \([a] = [a, \bar{a}], a \leq \bar{a} \),

\( \mathbb{IR}^n \): set of \( n \)-vectors with components from \( \mathbb{IR} \), interval vectors

\( \mathbb{IR}^{n \times n} \): set of \( n \)-by-\( n \) matrices with entries from \( \mathbb{IR} \). interval matrices

Elements from \( \mathbb{IR}^n \) and \( \mathbb{IR}^{n \times n} \) may be regarded as vector intervals and matrix intervals, respectively, w.r.t. the usual entrywise partial ordering, e.g.,

\[
[A] = ([a_{ij}])_{i,j=1}^n = ([a_{ij}, \bar{a}_{ij}])_{i,j=1}^n = [A, \bar{A}], \quad \text{where } A = (a_{ij})_{i,j=1}^n, \quad \bar{A} = (\bar{a}_{ij})_{i,j=1}^n.
\]

A vertex matrix of \([A]\) is a matrix \( A = (a_{ij})_{i,j=1}^n \) with \( a_{ij} \in \{a_{ij}, \bar{a}_{ij}\}, \)

\( i, j = 1, \ldots, n \).
A suitable partial order for the special class of matrices is the checkerboard order. For $A, B \in \mathbb{R}^{n \times n}$ define

$$A \leq^* B := (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \quad i, j = 1, 2, \ldots, n.$$ 

This partial order is related to the usual entry-wise partial order by

$$A \leq^* B \Leftrightarrow A^* \leq B^*, \text{ where } A^* := SAS, \ S := \text{diag}(1, -1, \ldots, (-1)^{n+1}),$$

is the checkerboard transformation.
A matrix interval \([A, \overline{A}]\) with respect to the usual entry-wise partial order can be represented as an interval \([\downarrow A, \uparrow A]^*\) with respect to the checkerboard order, where

\[
(\downarrow A)_{ij} := \begin{cases} 
    a_{ij} & \text{if } i + j \text{ is even,} \\
    \overline{a}_{ij} & \text{if } i + j \text{ is odd,}
\end{cases}
\]

\[
(\uparrow A)_{ij} := \begin{cases} 
    \overline{a}_{ij} & \text{if } i + j \text{ is even,} \\
    a_{ij} & \text{if } i + j \text{ is odd.}
\end{cases}
\]
Systems of linear interval equations \([A]x = [b]\)

Solution set 
\[ \Sigma := \Sigma ([A], [b]) := \{ x \in \mathbb{R}^n \mid Ax = b, \ A \in [A], \ b \in [b] \} \]

The matrix interval \([A]\) is called *regular* if \(A\) is nonsingular for all \(A \in [A]\).

Properties of the solution set

- \(\Sigma\) is closed.
- If \([A]\) is regular, then \(\Sigma\) is compact, connected, and convex in each orthant.
(Interval) Hull of the solution set

\[ [A]^H [b] := \square \Sigma ([A], [b]) \]

**Examples**

\[
\begin{pmatrix}
2 & 4 \\
-1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
x
\end{pmatrix} =
\begin{pmatrix}
-2 & 2 \\
-2 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
x
\end{pmatrix}
\]

Solution sets for Barth-Nuding and Hansen interval systems

**Examples (cont'd)**

\[
\begin{pmatrix}
2 & 3 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
x
\end{pmatrix} =
\begin{pmatrix}
0 & 120 \\
60 & 240
\end{pmatrix}
\]

Interval hulls for Barth-Nuding and Hansen interval systems
Important classes of matrices

An $n$-by-$n$ matrix $A$ is called

- **$M$-matrix** if $A$ can be written as $A = \alpha I - B$ for some nonnegative matrix $B$ and positive scalar $\alpha > \rho(B)$.
- **inverse $M$-matrix** if $A^{-1}$ exists and $A^{-1}$ is an $M$-matrix.
- **inverse nonnegative** if $A^{-1}$ exists and $0 \leq A^{-1}$.
- **positive (semi)-definite** if $A$ is symmetric and all principal minors of $A$ are positive (nonnegative).
• **sign regular (SR)** with signature \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) if all its minors of order \( k \) have sign \( \epsilon_k \) or are allowed also to vanish for all \( k = 1, \ldots, n \).

• **strictly sign regular (SSR)** with signature \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) if all its minors of order \( k \) are nonzero and have sign \( \epsilon_k \) for all \( k = 1, \ldots, n \).

• **almost strictly sign regular (ASSR)** with signature \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) if \( A \) is SR with signature \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) and any minor is nonzero if and only if the entries on the main diagonal of the corresponding submatrix are nonzero.

• **totally nonnegative (TN)** and **totally positive (TP)** if \( A \) is SR and SSR with signature \( \epsilon = (1, \ldots, 1) \), respectively.

• **totally nonpositive (t.n.p.)** and **totally negative (t.n.)** if \( A \) is SR and SSR with signature \( \epsilon = (-1, \ldots, -1) \), respectively.
Inverse nonnegative matrices

Examples of inverse nonnegative matrices

- **$M$-matrices.**

- Let $S = \text{diag} \left( 1, -1, \ldots, (-1)^{n-1} \right)$. Then for any nonsingular $SR$ matrix $A$ with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ such that $\epsilon_{n-1} \cdot \epsilon_n = 1$, $SAS$ is inverse nonnegative.

- Let $S = \text{diag} \left( 1, -1, \ldots, (-1)^{n-1} \right)$. Then for any nonsingular $SR$ matrix $A$ with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ such that $\epsilon_{n-1} \cdot \epsilon_n = -1$, $-SAS$ is inverse nonnegative.

**Proposition [Kuttler, 1971]**

Let $[A] = [A, \bar{A}]$ be a matrix interval and $A$ and $\bar{A}$ be inverse nonnegative. Then $[A]$ is inverse nonnegative and $\bar{A}^{-1} \leq A^{-1}$. 
Theorem [Beeck, 1974]

If $[A] \in \mathbb{IR}^{n \times n}$ is inverse nonnegative, then

$$A^H b = \begin{cases} 
\left[ A^{-1} b, \ A^{-1} \bar{b} \right] & \text{if } 0 \leq b, \\
\left[ A^{-1} b, \ A^{-1} \bar{b} \right] & \text{if } 0 \in [b], \\
\left[ A^{-1} b, \ \bar{A}^{-1} \bar{b} \right] & \text{if } \bar{b} \leq 0.
\end{cases}$$

In the general case, one has to solve at most $2n$ linear systems to find $\inf(A^H b)$ and similarly $\sup(A^H b)$. 
Interval Property

We say that a class $C$ of $n$-by-$n$ matrices possesses the interval property if for any $n$-by-$n$ interval matrix $[A] = [\underline{A}, \overline{A}] = ([a_{ij}, \overline{a}_{ij}])_{i,j=1,...,n}$ the membership $[A] \subseteq C$ can be inferred from the membership to $C$ of a specified set of its vertex matrices.
Classes of matrices possessing the interval property

- $M$-matrices or, more generally, inverse-nonnegative matrices [Kuttler, 1971], where only the bound matrices $A$ and $\bar{A}$ are required to be in the class;
- inverse $M$-matrices [Johnson and Smith, 2002], where all vertex matrices are needed;
- positive definite matrices [Bialas and Garloff, 1984], [Rohn, 1994], where a subset of cardinality $2^{n-1}$ is required (here only symmetric matrices in $[A]$ are considered).
In the following classes of matrices only $\downarrow A$ and $\uparrow A$ are needed:

- *SSR* matrices [Garloff, 1982], [Adm and Garloff].
- The following classes of matrices [Adm and Garloff, 2013], [Adm and Garloff]:
  - nonsingular *ASSR* matrices,
  - nonsingular tridiagonal *SR* matrices,
  - nonsingular totally nonnegative,
  - tridiagonal *TN* matrices,
  - nonsingular totally nonpositive.
If $\downarrow A$ and $\uparrow A$ are non-singular and totally nonnegative then the whole matrix interval $[\downarrow A, \uparrow A]^*$ is non-singular and totally nonnegative.
We denote by $\leq$ the lexicographic order on $\mathbb{N}^2$, i.e.,

$$(g, h) \leq (i, j) : \iff (g < i) \text{ or } (g = i \text{ and } h \leq j).$$

Set $E^\circ := \{1, \ldots, n\}^2 \setminus \{(1, 1)\}$, $E := E^\circ \cup \{(n + 1, 2)\}$.

Let $(s, t) \in E^\circ$. Then

$$(s, t)^+ := \min \{(i, j) \in E \mid (s, t) \leq (i, j), (s, t) \neq (i, j)\}.$$
Algorithm

Let $A \in \mathbb{R}^{n \times n}$. As $r$ runs in decreasing order over the set $E$, we define matrices $A^{(r)} = (a^{(r)}_{ij}) \in \mathbb{R}^{n \times n}$ as follows.

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s, t) \in E^\circ$:
   
   (a) if $a^{(r^+)}_{st} = 0$ then put $A^{(r)} := A^{(r^+)}$.
   
   (b) if $a^{(r^+)}_{st} \neq 0$ then put
   
   
   $$a^{(r)}_{ij} := \begin{cases} 
   a^{(r^+)}_{ij} - \frac{a^{(r^+)}_{it} a^{(r^+)}_{sj}}{a^{(r^+)}_{st}} & \text{for } i < s \text{ and } j < t, \\
   a^{(r^+)}_{ij} & \text{otherwise.}
   \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$ is called the matrix obtained from $A$ (by the Cauchon Algorithm).
Example

If \( n = 5 \) and \( A \) is totally positive, then

\[
\tilde{A} = \begin{bmatrix}
\end{bmatrix}
\]
Theorem [Goodearl, Launois and Lenagan, 2011], [Adm and Garloff, 2013]

- $A$ is totally nonnegative iff $0 \leq \tilde{A}$ and for all $i, j = 1, \ldots, n$
  \[ \tilde{a}_{ij} = 0 \implies \tilde{a}_{ik} = 0 \quad k = 1, \ldots, j - 1, \quad \text{or} \quad \tilde{a}_{kj} = 0 \quad k = 1, \ldots, i - 1. \]

\[
\tilde{A} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]

- If $A$ is totally nonnegative matrix then $A$ is nonsingular iff $0 < \text{diag}(\tilde{A})$. 

\[
\tilde{A} = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\end{bmatrix}
\]
Theorem [Adm and Garloff, 2013]

Let $A, B$ be nonsingular and totally nonnegative matrices and let $A \preceq^* Z \preceq^* B$. Then

1. $\tilde{A} \preceq^* \tilde{Z} \preceq^* \tilde{B}$;
2. $Z$ is nonsingular and totally nonnegative;
3. if $A, B$ possess the same pattern of zero minors then $Z$ has this pattern, too.
The assumption of nonsingularity of certain principal minors cannot be relaxed:

\[
A := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \lesssim^* Z := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \lesssim^* B := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

totally nonnegative \quad has a negative minor \quad totally nonnegative
Corollary [Adm and Garloff, 2013]

Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \preceq^* Z \preceq^* B$. If $A, B$ are totally nonnegative and $A[2,\ldots,n]$ and $B[2,\ldots,n]$ or $A[1,\ldots,n-1]$ and $B[1,\ldots,n-1]$ are nonsingular, then $Z$ is totally nonnegative, too.
Conjecture [Adm and Garloff]

Assume that $\downarrow A$ and $\uparrow A$ are nonsingular and SR matrices, then $[\downarrow A, \uparrow A]^*$ is nonsingular and SR?

A partial result

It was shown in [Garloff, 1996] that the conclusion is true if we consider instead of the two bound matrices a set of vertex matrices with the cardinality of at most $2^{2n-1}$ ($n$ being the order of the matrices).
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THANK YOU VERY MUCH