Matrix Methods for the Bernstein Form and their Application in Global Optimization

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Introduction

Solving optimization problems is of paramount importance in many real-life and scientific problems; polynomial global optimization problems form a significant part of them. One approach for their solution is based on the expansion of a polynomial into Bernstein polynomials, the so-called *Bernstein form*, see [1-5], [8]. This approach has the advantage that it does not require function evaluations which might be costly if the degree of the polynomial is high.

Shorthand notation for multi-indices is used; a multi-index $(i_1, \ldots, i_n)$ is abbreviated as i, where n is the number of variables. Comparison between and arithmetic operations with multi-indices are defined entry-wise. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, its monomials are defined as $x^i := \prod_{j=1}^{n} x_j^{i_j}$, and the abbreviations $\sum_{i=0}^{k} := \sum_{i_1=0}^{k_1} \ldots \sum_{i_n=0}^{k_n}$ and $\binom{k}{i} := \prod_{\alpha=1}^{n} \binom{k_\alpha}{i_\alpha}$ are used.

We will consider the unit box $\mathbf{u} := [0,1]^n$, since any compact nonempty box $\mathbf{x}$ of $\mathbb{R}^n$ can be mapped affinely upon $\mathbf{u}$. Let p be an n-variate polynomial of degree l which can be represented in the power form as $p(x) = \sum_{i=0}^{l} a_i x^i$. We expand p into Bernstein polynomials over $\mathbf{u}$ as

$$p(x) = \sum_{i=0}^{k} b_i^{(k)} B_i^{(k)}(x), \quad k \geq l, \tag{1}$$

where $B_i^{(k)}$ is the *i-th Bernstein polynomial of degree k*, $k \geq l$, defined as

$$B_i^{(k)}(x) = \binom{k}{i} x^i (1-x)^{k-i}. \tag{2}$$

The coefficients of this expansion are called the *Bernstein coefficients* of p over $\mathbf{u}$ and are given by

$$b_i^{(k)} = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}} a_j, \quad 0 \leq i \leq k. \tag{3}$$

The Bernstein coefficients can be organized in a multi-dimensional array $B(\mathbf{u}) = (b_i^{(k)})_{0 \leq i \leq k}$, the so-called *Bernstein patch*.

The Bernstein coefficients provide lower and upper bounds for the range of $p(x)$ over $\mathbf{u}$,

$$\min b_i^{(k)} \leq p(x) \leq \max b_i^{(k)}, \quad \text{for all } x \in \mathbf{u}. \tag{4}$$

Equality holds in the left or right inequality in (4) if and only if the minimum or the maximum, respectively, is attained at a vertex of $\mathbf{u}$, i.e., if $i_j \in \{0, k_j\}$, $j = 1, \ldots, n$.

We can improve the enclosure for the range of p given by (4) by elevating the degree k of the Bernstein expansion or by subdividing $\mathbf{u}$. The subdivision is more efficient than the degree elevation.

From the Bernstein coefficients $b_i^{(k)}$ of p over $\mathbf{u}$, we can compute by the de Casteljau algorithm the Bernstein coefficients over sub-boxes $\mathbf{u}_1$ and $\mathbf{u}_2$ resulting from subdividing $\mathbf{u}$ in the s-th direction, i.e.,

$$\begin{aligned} \mathbf{u}_1 &:= [0,1] \times \ldots \times [0,\lambda] \times \ldots \times [0,1], \\ \mathbf{u}_2 &:= [0,1] \times \ldots \times [\lambda,1] \times \ldots \times [0,1], \end{aligned} \tag{5}$$

for some $\lambda \in (0,1)$.

Bounding the range of a function over a box is an important task in global optimization when a branch and bound approach is applied. In the case that the optimization problem is convex we have the advantage that each local minimum is also a global one. Therefore, it is useful to know when a function is convex over a box. A well-known criterion for convexity is that the Hessian matrix is positive definite.

Main results

In our talk we present the following results:

- We propose a new method for the computation of the Bernstein coefficients of multivariate Bernstein polynomials which involves matrix operations such as multiplication and transposition and which is more efficient than the matrix method presented in [6].

- We present a new method for the calculation of the Bernstein coefficients over a sub-box by premultiplying the matrix representing the Bernstein patch by matrices which depend on the intersection point λ.

- As an application to global optimization, we propose a test for the convexity of a polynomial p. This check depends on the *interval Hessian matrix* that is obtained by the entry-wise application of the range enclosure property (4). Following [7], we test the positive semidefiniteness of this interval matrix which leads to the test for convexity of p.

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