A linear iterative interval method for computing the generalized inverse of an matrix

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Introduction

There are three principal situations in which it is required to obtain numerically a generalized inverse of a given matrix [1]:

(i) the case in which any $\{1\}$-inverse will suffice;

(ii) the cases in which any $\{1,3\}$-inverse (or sometimes any $\{1,4\}$-inverse) will do; and

(iii) the case in which a $\{2\}$-inverse having a specified range and null space is required.

The inverse desired in case (iii) is, in the majority of cases, the Moore–Penrose inverse, which is the unique $\{2\}$-inverse of the given matrix A having the same range and null space as A^*. The Drazin inverse can also be fitted into this pattern, being the unique $\{2\}$-inverse of A having the same range and null space as A^l, where l is any integer not less than the index of A. When $l = 1$, this is the group inverse. Generalized inverses are closely associated with linear equations, orthonormalization, least-squares solutions, singular values, and various matrix factorizations. In particular, the QR-factorization and the Singular

Value Decomposition (SVD) figure prominently in the computation of the Moore–Penrose inverse.

Zhan et al. [2] suggested an interval iterative method for computing Moore–Penrose inverse of the full row (or column) rank matrix. Motivated by this work, here, we will attempt to introduce a new linear iterative interval method for computing the $\{1\}$-inverse of a given arbitrary matrix A. It should be noted that the given matrix A is not itself an interval matrix, but the method is interval to enclosing its interval inverse as sharp as possible, like interval Newton's method [3] which can enclose any root of a given real nonlinear equation.

Main Results

In this section, we try to extend a new linear iterative of interval sequence, say $\{\mathbf{X}^{(k)}\}$, converging to $\{1\}$-inverse, denoted by $A^{\dagger}$, of a given matrix A. We recall that the order of convergence of such an iteration is defined by its corresponding *residuals* rate

$$\mathbf{R}^{(k)} = P_{R(A)} - A\mathbf{X}^{(k)}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where converges to $\mathbf{0}$ as $k \to \infty$, or $\mathbf{X}^{(k)} \to A^{\dagger}$. The iterative residual (1) is linear if there is a positive constant M such that

$$d(\mathbf{R}^{(k+1)}) \leq M\, d(\mathbf{R}^{(k)}), \quad k = 0, 1, 2, \ldots.$$

Suppose that $\rho(\mathbf{R}^{(0)}) = \rho(P_{R(A)} - A\mathbf{X}^{(0)}) < 1$. We now consider the following interval extension of iterative method for computing $\{1\}$-inverse

$$\mathbf{X}^{(k+1)} = \left(\mathbf{X}_c^{(k)} + \mathbf{X}_c^{(0)}(I - A\mathbf{X}^{(k)})\right) \cap \mathbf{X}^{(k)}, k = 0, 1, 2, \ldots, \tag{2}$$

where $A \in C^{m\times n}$, $\mathbf{X}_c^{(0)} \in R(A^*, A^*)$, i.e., $A^*BA^* \subset \mathbf{X}^{(0)}$ for some $B \in C^{m\times n}$, and $\mathbf{X}_c^{(k)} = \mathrm{mid}(\mathbf{X}^{(k)})$. Evidently, $\mathbf{X}^{(k+1)} \subset \mathbf{X}^{(k)}$, and because of inclusion monotonicity, $A^{\dagger} = \lim \mathbf{X}_c^{(k)} \in \mathbf{X}^{(k)}$ as $k \to \infty$.

Taking into account (2), we can write

$$\begin{aligned}\mathbf{X}^{(k+1)} &= \left(\mathbf{X}_c^{(k)} + \mathbf{X}_c^{(0)}(\mathbf{R}^{(k)})\right) \cap \mathbf{X}^{(k)} \\ &= \left(\mathbf{X}_c^{(k)} + \mathbf{X}_c^{(0)}(P_{R(A)} - A\mathbf{X}^{(k)})\right) \cap \mathbf{X}^{(k)}, \quad k = 0, 1, \ldots. \end{aligned} \tag{3}$$

Consequently,

$$\begin{aligned}\mathbf{R}^{(k+1)} &= \left(P_{R(A)} - \mathbf{X}^{(k+1)}\right) \cap \mathbf{R}^{(k)} \\ &= \left(P_{R(A)} - A\mathbf{X}_c^{(k)} - A\mathbf{X}_c^{(0)}\mathbf{R}^{(k)}\right) \cap \mathbf{R}^{(k)} \\ &= \left(\mathbf{R}_c^{(k)} - A\mathbf{X}_c^{(0)}\mathbf{R}^{(k)}\right) \cap \mathbf{R}^{(k)}, \quad k = 0, 1, \ldots. \end{aligned} \tag{4}$$

It follows from (4) that

$$d(\mathbf{R}^{(k+1)}) \leq Md(\mathbf{R}^{(k)}), \tag{5}$$

for some $M \geq 0$. Since we have assumed that $\rho(\mathbf{R}^{(0)}) < 1$, the sequence of residuals converges to the zero matrix as k approaches to infinity. Accordingly, the sequence (1) converges. Finally, we have $A\mathbf{R}^{(\infty)} = P_{R(A)}$, since $P_{R(A)} - d(\mathbf{R}^{(k)}) \to O$, as $k \to \infty$. In particular, we have proved $\mathbf{R}^{(\infty)}$ is a $\{1\}$-inverse of A.

To obtain a second order convergence of interval iterative method, it suffices to consider the following modification of method (2)

$$\mathbf{X}^{(k+1)} = \left(\mathbf{X}_c^{(k)} + \mathbf{X}_c^{(k)}(I - A\mathbf{X}^{(k)})\right) \cap \mathbf{X}^{(k)}, k = 0, 1, 2, \ldots, \tag{6}$$

Although this modification increases convergence rate from one to two, however, we need to do more computation per iterate.

To sum up, we have claimed that we have developed a new method for computing an interval sequence that includes generalized inverses of a given rectangular matrix. Our method seems to be linear, and developing it to higher convergence method is straightforward. We hope this study shed new light on this field of linear algebra which there is little research devoted to it. Based on our best knowledge, there was only paper in this field and we have cited it here, [2]. We would be grateful if someone knows about this topic and inform us. We believe that much remains to be done in the area of interval methods for generalized inverses.

References

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