Permuted graph bases for verified computation of invariant subspaces

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The problem

A subspace $U \in \mathbb{C}^{n\times k}$ is called *invariant* under $H \in \mathbb{C}^{n\times n}$ if Hu is in U for all u in U [1]. The invariant subspace problem can be stated as finding $U \in \mathbb{C}^{n\times k}$ and $R \in \mathbb{C}^{k\times k}$ such that $HU = UR$, and the eigenvalues of R are a specified subset of those of H.

If one constrains U to be in the form $U = \begin{bmatrix} I_{k\times k} \\ X_{(n-k)\times k} \end{bmatrix}$, the problem of finding an invariant subspace can be recast as a *non-Hermitian algebraic Riccati equation* (NARE)

$$F(X) := Q + XA + \tilde{A}X - XGX = 0, \tag{1}$$

where $H = \begin{bmatrix} A_{k\times k} & -G_{k\times(n-k)} \\ -Q_{(n-k)\times k} & -\tilde{A}_{(n-k)\times(n-k)} \end{bmatrix}$ and $R = A - GX$.

Permuted graph bases

An idea reappeared recently in the matrix equation community [3] is that by applying a suitable permutation of the entries one can get an equation in which the solution X has smaller entries.

Theorem 1 ([2]). *Let $U \in \mathbb{C}^{n\times k}$ have full column rank. Then, there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ so that the top $k\times k$ submatrix E of $P^T U = \begin{bmatrix} E \\ A \end{bmatrix}$ is nonsingular, and the matrix $Z = AE^{-1} \in \mathbb{R}^{(n-k)\times k}$ is such that $|Z_{ij}| \leq 1$ for all i, j.*

By constructing this permutation P, we can replace the original NARE (1) with the one associated with $\tilde{H} = PHP^T$, whose solution Z has smaller entries.

An efficient enclosure for the solutions to NAREs

From now on we focus on the NARE(1). We wish to use the following classical result to find an enclosure for the solution X.

Theorem 2 ([4] (modified Krawczyk method)). *Assume that $f : D \subset \mathbb{C}^N \to \mathbb{C}^N$ is continuous in D. Let $\tilde{x} \in D$ and $\mathbf{z} \in \mathbb{IC}^N$ be such that $\tilde{x} + \mathbf{z} \subseteq D$. Moreover, assume that $\mathcal{S} \subset \mathbb{C}^{N\times N}$ is a set of matrices containing all slopes $S(\tilde{x}, y)$ for $y \in \tilde{x} + \mathbf{z} := \mathbf{x}$. Finally, let $R \in \mathbb{C}^{N\times N}$. Denote by $\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S})$ the set*

$$\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) := \{-Rf(\tilde{x}) + (I - RS)z : S \in \mathcal{S}, z \in \mathbf{z}\}.$$

Then, if

$$\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \operatorname{int} \mathbf{z},$$

the function f has a zero x^* in $\tilde{x} + \mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \mathbf{x}$. Moreover, if $\mathcal{S}$ also contains all slope matrices $S(x, y)$ for $x, y \in \mathbf{x}$, then this zero is unique in $\mathbf{x}$.

The recent works [5, 6] have successfully applied the modified Krawczyk method to several matrix equations, adding some crucial issues:

1. Let
$$A - GX = V_1\Lambda_1 W_1; \text{ with } V_1, W_1, \Lambda_1 \in \mathbb{C}^{k\times k},$$
$$\Lambda_1 = \operatorname{Diag}(\lambda_{11}, \ldots, \lambda_{k1}), V_1 W_1 = I,$$

and

$$\tilde{A}^* - G^* X^* = V_2 \Lambda_2 W_2; \text{ with } V_2, W_2, \Lambda_2 \in \mathbb{C}^{(n-k)\times(n-k)},$$

$$\Lambda_2 = \mathrm{Diag}(\lambda_{12}, \ldots, \lambda_{(n-k)2}), V_2 W_2 = I.$$

Then, set

$$R = (V_1^{-T} \otimes W_2^*).\Delta^{-1}.(V_1^T \otimes W_2^{-*}), \text{ where } \Delta = I \otimes \Lambda_2^* + \Lambda_1^T \otimes I.$$

This choice of R is so that its computation can be performed in $O(n^3)$, rather than the $O(n^5)$ obtained by vectorization without this improvement.

2. To reduce the problematic *wrapping effect* of interval arithmetic, use $\hat{f}$ as a linearly transformed function instead of f

$$\hat{f}(\hat{x}) = (V_1^T \otimes W_2^{-*}) f((V_1^{-T} \otimes W_2^*)\hat{x}),$$

where $(V_1^{-T} \otimes W_2^*)\hat{x} = x$.

We combine ideas from these two approaches to obtain an algorithm that can find enclosures for a larger class of problems in our experiments. A suitable modification of the ideas in Theorem 1 [3] can be used to work with structured invariant subspace problems and Hermitian algebraic Riccati equations (CAREs).

References

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