On cores of weakly oligomorphic relational structures

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joint work with Ch. Pech

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A relational structure **A** is called *weakly oligomorphic* if for every arity there are finitely many relations that can be defined by sets of positive existential formulæ.

Proposition

A countable structure **A** is weakly oligomorphic if and only if End(A) is oligomorphic (i.e. there are just finitely many invariant relations of End(A) of any arity).

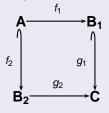
A local homomorphism of a structure **A** is a homomorphism from a finite substructure of **A** to **A**.

Definition (Cameron, Nešetřil 2002)

A structure **A** is **homomorphism-homogeneous** if every local homomorphism of **A** extends to an endomorphism of **A**.

Homo-almagamation property of a class $\ensuremath{\mathcal{C}}$ of finite relational structures

If $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, $f_1 : \mathbf{A} \mapsto \mathbf{B_1}$ is a homomorphism, and $f_2 : \mathbf{A} \hookrightarrow \mathbf{B_2}$ is an embedding, then there are $\mathbf{C} \in \mathcal{C}$, an embedding $g_1 : \mathbf{B_1} \hookrightarrow \mathbf{C}$, and a homomorphism $g_2 : \mathbf{B_2} \mapsto \mathcal{C}$ such that the following diagram commutes:



i.e. $g_1 \circ f_1 = g_2 \circ f_2$.

Characterization of the ages of countable homomorphism-homogeneous relational structures

Theorem

- (a) The age of any homomorphism-homogeneous structure has property HAP.
- (b) If a class C of finite relational structures
 - is closed under isomorphism,
 - has only countably many isomorphism types, and
 - has properties HP, JEP and HAP,

then there is a countable homomorphism-homogeneous relational structure whose age is C.

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A class of finite relational structures over the same signature, that is closed under isomorphism and that has properties HP, JEP, and HAP is called **hom-almagamation class**.

A structure **A** is **weakly homomorphism-homogeneous** if whenever **B** < **C** are finite substructures of **A**, then every homomorphism $f : \mathbf{B} \rightarrow \mathbf{A}$ extends to **C**.

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Remark

A countable structure is weakly homomorphism-homogeneous iff it is homomorphism-homogeneous.

Let **H** and **H**' be two relational structures. We write $\mathbf{H} \leq_h \mathbf{H}'$ if

- $Age(H) \supseteq Age(H')$, and
- for all finite A ≤ B ≤ H we have that every homomorphism from A to H' extends to a homomorphism from B to H'.

Ages and weak homomorphism-homogeneity

Proposition

Let **H** and **H**' be two relational structures.

- If H ≤_h H', and H is weakly homomorphism-homogeneous, then H' is weakly homomorphism-homogeneous, too.
- 2 If H' is weakly homomorphism-homogeneous, and Age(H) = Age(H'), then $H \leq_h H'$.

Corollary

Let **A**, and **B** be two weakly homomorphism-homogeneous structures with the same age. Then $\mathbf{A} \leq_h \mathbf{B}$ and $\mathbf{B} \leq_h \mathbf{A}$. In particular, any two countable homomorphism-homogeneous relational structures with the same age are homomorphism-equivalent.

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A relational structure is a *core*, if its all endomorphisms are embeddings.

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Let C be a class of relational structures over the same signature and let $\mathbf{A} \in C$. We say that \mathbf{A} is *h*-irreducible if for every $\mathbf{B} \in C$ and every homomorphism $f : \mathbf{A} \to \mathbf{B}$ holds that f is an embedding.

For a given relational structure **A**, the class of all finite structures of the same type like **A** that are hom-irreducible in the age of **A** will be denoted by C_A .

On the class of h-irreducible structures

Let \mathcal{A}, \mathcal{B} be classes of rel. structures over a common signature. \mathcal{A} **projects** onto \mathcal{B} ($\mathcal{A} \to \mathcal{B}$) if

$$(\forall A \in \mathcal{A})(\exists B \in \mathcal{B}) \quad A \to B$$

- Let C be a hom-amalgamation class, let D be the class of all structures from C that are hom-irreducible in C.
 If C → D, then D is a Fraïssé class.
- If A is a weakly oligomorphic and weakly homomorphism-homogeneous relational structure, then

$$\operatorname{Age}(\mathsf{A}) \to \mathcal{C}_{\mathsf{A}}$$

Cores and HH structures

Proposition

Let A be a countable homomorphism-homogeneous relational structure, such that $Age(A) \rightarrow C_A$. Then A has a core C with age C_A .

Corollary

Every countable weakly oligomorphic homomorphism-homogeneous relational structure **A** has a core **C** with age C_A .

Theorem

Let **A** be a countable weakly oligomorphic homomorphism-homogeneous relational structure. Then **A** contains a substructure **F** that is isomorphic to the Fraïssé limit of C_A . Moreover, **F** and **A** are hom-equivalent, and **F** is oligomorphic.

Corollary

Every countable weakly oligomorphic homomorphism-homogeneous relational structure **A** contains, up to isomorphism, a unique hom-equivalent homomorphism-homogeneous core **F**. Moreover, **F** is oligomorphic and homogeneous.

Theorem

Let **A** be a countable weakly oligomorphic relational structure. Then **A** is hom-equivalent to a finite or ω -categorical structure **F**. Moreover, **F** embedds into **A**.