The reducts of the homogeneous C-relation, and tractable phylogeny problems

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Outline

1. Phylogeny problems
2. Homogeneous C-relations
3. Universal algebra and Ramsey theory
4. Generalizations for all ω-categorical structures?

Part 1: Phylogenetic Reconstruction

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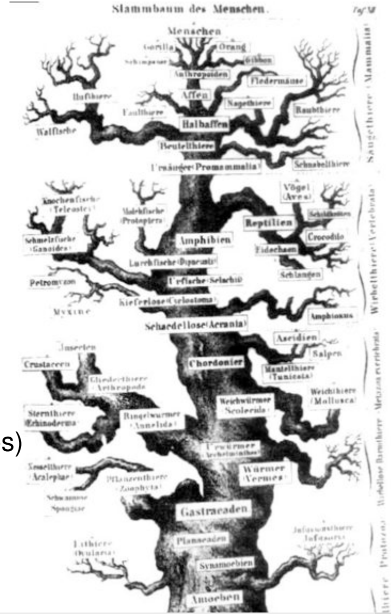

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- There are about 100 million species presently living on earth.
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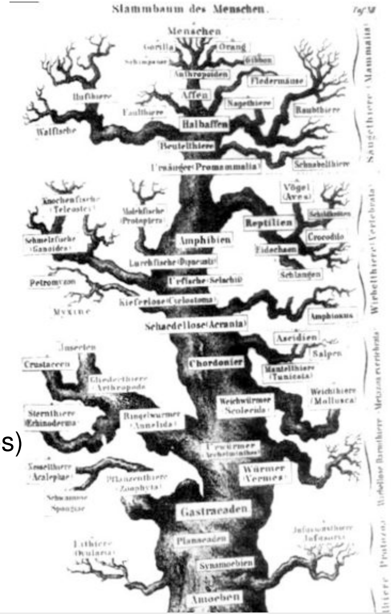
- Computationally challenging
  - Large amounts of data
  - Conflicting information



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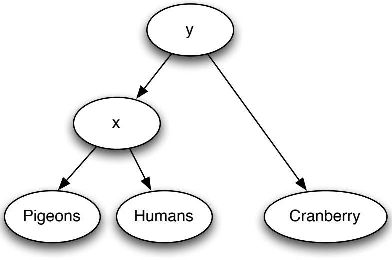

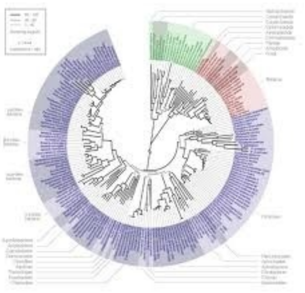

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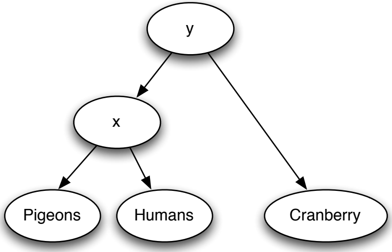

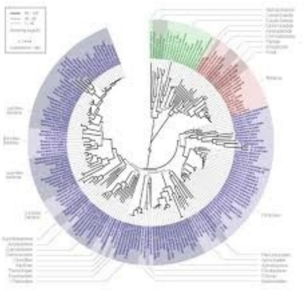

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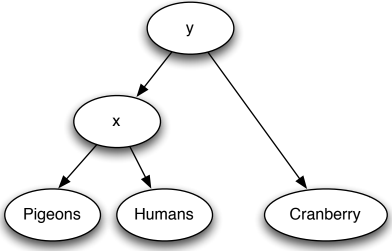

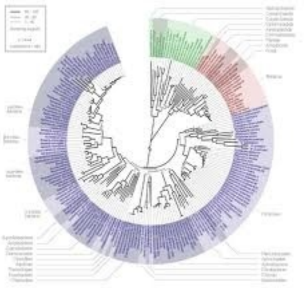

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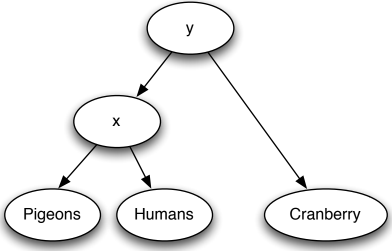

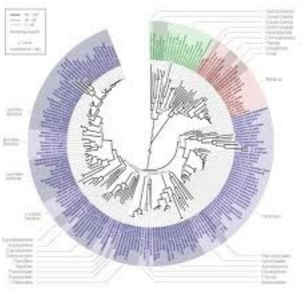

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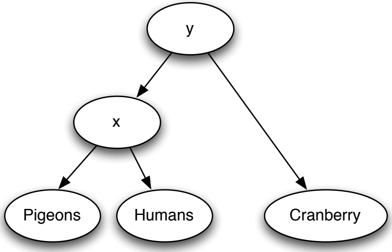

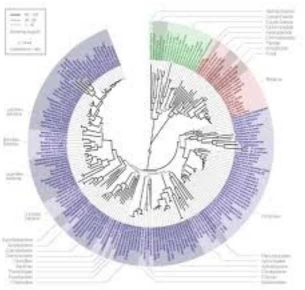

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- Example: pigeons humans | cranberries

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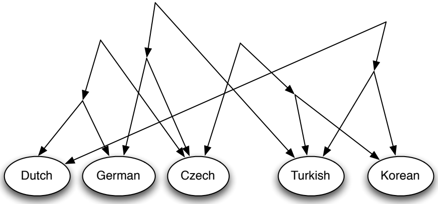

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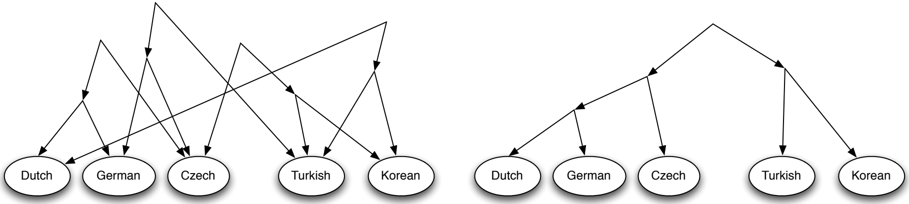

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- Work of Aho, Sagiv, Szymanski, Ullman independently motivated in database theory
- Running time improved to $O(n^{3/2})$ by Henzinger+King+Warnow'95, and to $O(n \log^2 n)$ by Holm+deLichtenberg+Thorup'98.

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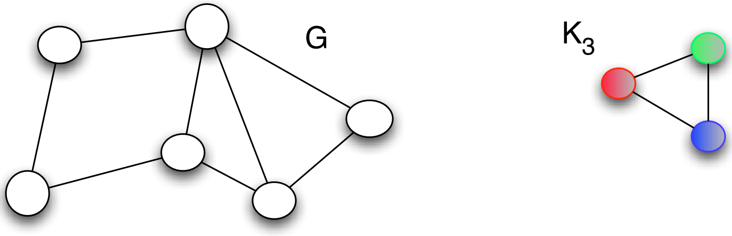
Example: 3-colorability is CSP(K_3)



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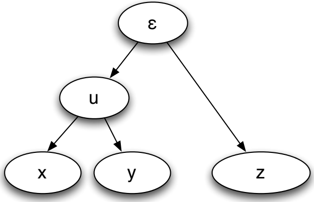

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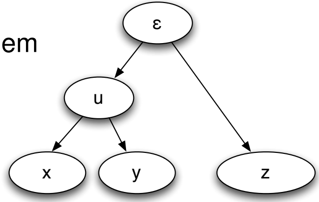

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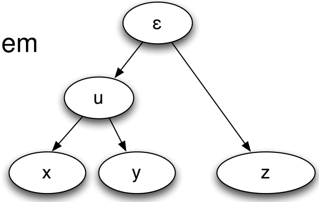

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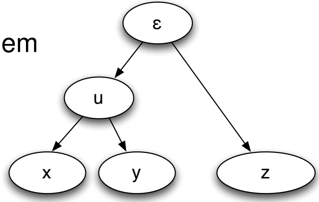

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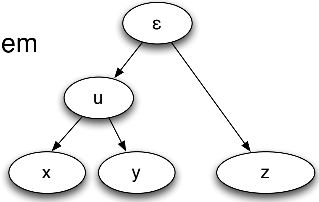

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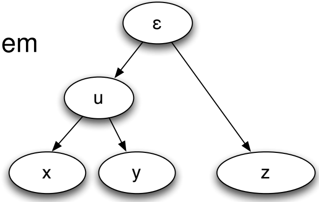

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  - C5 $\forall a, b.\, \exists e. C(e; a, b)$.
- $\text{Aut}(\mathbb{L}; C)$ is a Jordan permutation group
- Literature on C-minimal structures (in analogy to *o*-minimal structures, where the role of the order is played by a *C*-relation)

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Input: A set of variables V, a set of quartets $Q \subseteq V^4$.

Question: Is there a tree T with leaves V such that for each $(x, y, u, v) \in Q$ the shortest path from x to y does not intersect the shortest path from u to v?

Template: $\big(\mathbb{L}; \{(x, y, u, v) \ : \ (xy|u \wedge xy|v) \vee (x|uv \wedge y|uv)\}\big)$

Tree Balance Constraints

Input: A set of variables V, a set of quartets $Q \subseteq V^4$.

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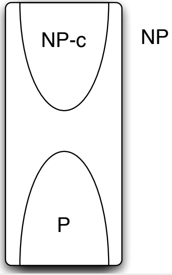

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- Using the Ramsey techniques that will be presented in this talk.

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Key ingredient in proof: Ramsey theory

Ramsey Classes

Write $\binom{G}{H}$ for the set of all induced substructures of G that are isomorphic to H.

Definition

For structures G, H, P, write

$$G \rightarrow (H)^P_k$$

if for all $\chi \colon \binom{G}{P} \rightarrow [k]$ there exists $H' \in \binom{G}{H}$ such that χ is constant on $\binom{H'}{P}$.

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Example: The class of all finite linearly ordered graphs. (Nešetřil-Rödl)

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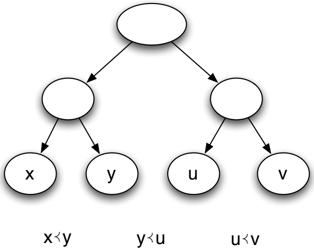

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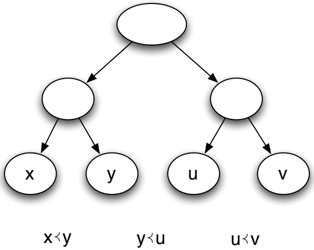

Theorem (consequence of Miliken'79).
The class of all finite structures that embed into $(\mathbb{L}; C, \prec)$ is a Ramsey class.

- A consequence of Miliken'79, generalizing earlier results of Deuber.
- See B.+Piguet'09 for a direct proof.

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Remarks.

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- If Δ_1 and Δ_2 have finite relational signature, then there are finitely many canonical behaviors.

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In the same way we can characterize primitive positive definability by replacing with (canonical) polymorphisms.

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Fact 2 [B+Pinsker+Tsankov'11]:
open subgroups of extremely amenable groups are extremely amenable
(combinatorial counterpart: expansions of homogeneous structures by finitely many constants preserve the Ramsey property)

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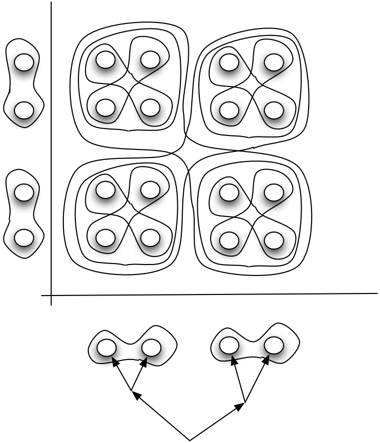

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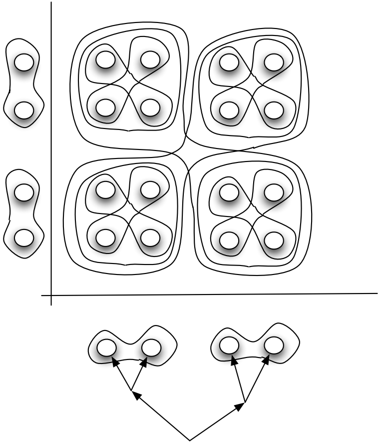

Example: the tree balance relation $B(x, y, u, v)$ defined by $xy|uv \vee xu|yv \vee xv|yu$ is preserved by the affine tree polymorphism.

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        let s be a non-constant solution
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In all those cases, dichotomy coincides with a complexity dichotomy NPc/P

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- Let G be an oligomorphic permutation group. Does G always have an extremely amenable oligomorphic subgroup?