

Better Bounds for Incremental Frequency Allocation in Bipartite Graphs

Marek Chrobak^a, Łukasz Jeż^{b,c}, Jiří Sgall^{d,*}

^aDepartment of Computer Science, Univ. of California, Riverside, CA 92521, US

^bInstitute of Computer Science, ul. Joliot-Curie 15, University of Wrocław, 50-383 Wrocław, Poland.

^cInstitute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

^dComputer Science Institute of Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, CZ-11800 Praha 1, Czech Republic.

Abstract

We study frequency allocation in wireless networks. A wireless network is modeled by an undirected graph, with vertices corresponding to cells. In each vertex we have a certain number of requests, and each of those requests must be assigned a different frequency. Edges represent conflicts between cells, meaning that frequencies in adjacent vertices must be different as well. The objective is to minimize the total number of used frequencies.

The offline version of the problem is known to be NP-hard. In the incremental version, requests for frequencies arrive over time and the algorithm is required to assign a frequency to a request as soon as it arrives. Competitive incremental algorithms have been studied for several classes of graphs. For paths, the optimal (asymptotic) ratio is known to be $4/3$, while for hexagonal-cell graphs it is between 1.5 and 1.9126. For ξ -colorable graphs, the ratio of $(\xi + 1)/2$ can be achieved.

In this paper, we prove nearly tight bounds on the asymptotic competitive ratio for bipartite graphs, showing that it is between 1.428 and 1.433. This improves the previous lower bound of $4/3$ and upper bound of 1.5. Our proofs are based on reducing the incremental problem to a purely combinatorial (equivalent) problem of constructing set families with certain intersection properties.

Keywords: online algorithms, frequency allocation, graph algorithms

1. Introduction

Static frequency allocation. In the frequency allocation problem, we are given a wireless network and a collection of requests for frequencies. The network is modeled by a (possibly infinite) undirected graph G , whose vertices correspond to the network's cells. Each request is associated with a vertex, and requests in the same vertex must be assigned different frequencies. Edges represent conflicts between cells, meaning that frequencies in adjacent vertices must be different as well. The objective is to minimize the total number of used frequencies. We will refer to this model as *static*, as it corresponds to the scenario where the set of requests in each vertex does not change over time.

A more rigorous formulation of this static frequency allocation problem is as follows: Denote by ℓ_v the *load* (or *demand*) at a vertex v of G , that is the number of frequency requests at v . A frequency allocation is a function that assigns a set L_v of frequencies (represented, say, by positive integers) to each vertex v and satisfies the following two conditions: (i) $|L_v| = \ell_v$ for each vertex v , and (ii) $L_v \cap L_w = \emptyset$ for each edge $\{v, w\}$. The total number of frequencies used is $|\bigcup_{v \in G} L_v|$, and this is the quantity we wish to minimize. We will use notation $opt(G, \vec{\ell})$ to denote the minimum number of frequencies for a graph G and a demand vector $\vec{\ell}$.

*Corresponding author

Email addresses: marek@cs.ucr.edu (Marek Chrobak), lje@cs.uni.wroc.pl (Łukasz Jeż), sgall@iuuk.mff.cuni.cz (Jiří Sgall)

If one request is issued per node, then $\text{opt}(G, \bar{\ell})$ is equal to the chromatic number of G , which immediately implies that the frequency allocation problem is NP-hard. In fact, McDiarmid and Reed [1] have shown that the problem remains NP-hard for the graph representing the network whose cells are regular hexagons in the plane, which is a commonly studied abstraction of wireless networks. (See, for example, the surveys in [2, 3].) Polynomial-time $\frac{4}{3}$ -approximation algorithms for this case appeared in [1] and [4].

Incremental frequency allocation. In the incremental version of frequency allocation, requests arrive over time and an incremental algorithm must assign frequencies to requests as soon as they arrive. An incremental algorithm \mathcal{A} is called *asymptotically R -competitive* if, for any graph G and load vector $\bar{\ell}$, the total number of frequencies used by \mathcal{A} is at most $R \cdot \text{opt}(G, \bar{\ell}) + \lambda$, where λ is a constant independent of $\bar{\ell}$. We allow λ to depend on the class of graphs under consideration, in which case we say that \mathcal{A} is R -competitive for this class. We refer to R as the *asymptotic competitive ratio* of \mathcal{A} . As in this paper we focus only on the asymptotic ratio, we will skip the word “asymptotic” (unless ambiguity can arise), and simply use terms “ R -competitive” and “competitive ratio” instead. Following the terminology in the literature (see [5, 6]), we will say that the competitive ratio is *absolute* when the additive constant λ is equal 0.

Naturally, research in this area is concerned with designing algorithms with small competitive ratios for various classes of graphs, as well as proving lower bounds. For hexagonal-cell graphs, Chan *et al.* [5, 6] found an incremental algorithm with competitive ratio 1.9216 and proved that no ratio better than 1.5 is possible. A lower bound of $4/3$ for paths was given in [7], and later Chrobak and Sgall [8] gave an incremental algorithm with the same ratio. Paths are in fact the only non-trivial graphs for which tight asymptotic ratios are known. As pointed out earlier, there is a strong connection between frequency allocation and graph coloring, so one would expect that the competitive ratio can be bounded in terms of the chromatic number. Indeed, for ξ -colorable graphs Chan *et al.* [5, 6] gave an incremental algorithm with competitive ratio of $(\xi + 1)/2$. (This ratio is in fact absolute.) On the other hand, the best known lower bounds on the competitive ratio, 1.5 in the asymptotic and 2 in the absolute case [5, 6], hold for hexagonal-cell graphs (for which $\xi = 3$), but no stronger bounds are known for graphs of higher chromatic number.

We remark that computing the optimal offline solution for the incremental instance is equivalent to computing the optimal solution to the static instance where the request set consists of all requests in the incremental request sequence.

Our contribution. We prove nearly tight bounds on the optimal competitive ratio of incremental algorithms for bipartite graphs (that is, for $\xi = 2$), showing that it is between $10/7 \approx 1.428$ and $(18 - \sqrt{5})/11 \approx 1.433$. This improves the lower and upper bounds for this version of frequency allocation. The best previously known lower bound was $4/3$, which holds in fact even for paths [7, 8]. The best upper bound of 1.5 was shown in [5, 6] and it holds even in the absolute case.

Our proofs are based on reducing the incremental problem to a purely combinatorial (equivalent) problem of constructing set families, which we call *F-systems*, with certain intersection properties. A rather surprising consequence of this reduction is that the optimal competitive ratio can be achieved by an algorithm that is topology-independent; it assigns a frequency to each vertex v based only on the current optimum value, the number of requests to v , and the partition to which v belongs; that is, independently of the frequencies already assigned to the neighbors of v .

To achieve a competitive ratio below 2 for bipartite graphs, we need to use frequencies that are shared between the two partitions of the graph. The challenge is then to assign these shared frequencies to the requests in different partitions so as to avoid collisions—in essence, to break the symmetry. In our construction, we develop a symmetry-breaking method based on the concepts of “collisions with the past” and “collisions with the future”, which allows us to derive frequency sets in a systematic fashion.

Our work is motivated mostly by theoretical interest, as there is no reason why realistic wireless networks would form a bipartite graph. There is an intuitive connection between the chromatic number of a graph and optimal frequency allocation, and exploring the exact nature of this connection is worthwhile and challenging. Our results constitute a significant progress for the case of two colors, and we believe that some ideas from this paper—the concept of F-systems and our symmetry-breaking method, for example—can be extended to frequency assignment problems on graphs with larger chromatic number.

Other related work. Determining optimal absolute ratios is usually easier than for asymptotic ratios and

it has been accomplished for various classes of graphs, including paths [7], general bipartite graphs [5, 6], hexagonal-cell graphs and general 3-colorable graphs [5, 6]. The asymptotic ratio model, however, is more relevant to practical scenarios where the number of frequencies is typically very large, so the additive constant can be neglected.

In the *dynamic version* of frequency allocation each request has an arrival and departure time. At each time, any two requests that have already arrived but not departed and are in the same or adjacent nodes must be assigned different frequencies. As before, we wish to minimize the total number of used frequencies. As shown by Chrobak and Sgall [8], this dynamic version is NP-hard even for the special case when the input graph is a path.

It is natural to study the online version of this problem, where we introduce the notion of “time” that progresses in discrete steps, and at each time step some requests may arrive and some previously arrived requests may depart. This corresponds to real-life wireless networks where customers enter and leave a network’s cells over time, in an unpredictable fashion. An online algorithm needs to assign frequencies to requests as soon as they arrive. The competitive ratio is defined analogously to the incremental case. (The incremental version can be thought of as a special case in which all departure times are infinite.) This model has been well studied in the context of job scheduling, where it is sometimes referred to as time-online. Very little is known about this online dynamic case. Even for paths the optimal ratio is not known; it is only known that it is between $\frac{14}{9} \approx 1.571$ [8] and $\frac{5}{3} \approx 1.667$ [7].

2. Preliminaries

For concreteness, we will assume that frequencies are identified by positive integers, although it does not really matter. Recall that we use the number of frequencies as the performance measure. In some literature [7, 9, 6], authors used the maximum-numbered frequency instead. It is not hard to show (see [8]) that these two approaches are equivalent¹.

For a bipartite graph $G = (A, B, E)$, it is easy to characterize the optimum value. As observed in [7, 8], in this case the optimum number of frequencies is

$$\text{opt}(G, \bar{\ell}) = \max_{\{u,v\} \in E} \{\ell_u + \ell_v\}. \quad (1)$$

For completeness, we include a simple proof: Trivially, $\text{opt}(G, \bar{\ell}) \geq \ell_u + \ell_v$ for each edge $\{u, v\}$. On the other hand, denoting by ω the right-hand side of (1), we can assign frequencies to nodes as follows: for $u \in A$, assign to u frequencies $1, 2, \dots, \ell_u$, and for $u \in B$ assign to u frequencies $\omega - \ell_u + 1, \omega - \ell_u + 2, \dots, \omega$. This way each vertex u is assigned ℓ_u frequencies and no two adjacent nodes share the same frequency.

Throughout the paper, we will use the convention that if $c \in \{A, B\}$, then c' denotes the partition other than c , that is $\{c, c'\} = \{A, B\}$.

3. An Example

We now examine a simple example that illustrates why an online algorithm may be forced to use more frequencies than an offline optimum. The graph G is a path of length 4, with vertices v_1, v_2, v_3, v_4 , in this order (see Figure 1). Let’s say that in the first six steps we issue three requests to v_1 and three requests to v_4 .

If we use 5 or more frequencies then the request sequence can end here, and the ratio will be $5/3$, since in the optimal solution the requests on v_1 and on v_4 can be assigned the same frequencies.

The other case is that we use no more than 4 frequencies so far. This implies that v_1 and v_4 share at least 2 frequencies. Suppose that then the request sequence continues with three requests to v_2 and three requests to v_3 . All these six requests must be assigned different frequencies, and these frequencies must also be different from those two shared by v_1 and v_4 . As a result, we are forced to use 8 frequencies in total.

¹The proof in [8], however, involves a transformation of the algorithm that makes it *not* topology independent.

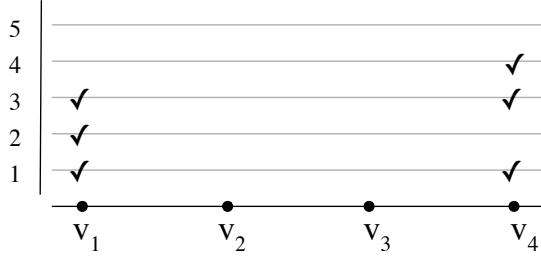


Figure 1: An example of frequency allocation for a path of length 4, after three requests to v_1 and three requests to v_2 . Frequencies 1,2,3 are assigned to v_1 and frequencies 1,3,4 are assigned to v_4 .

The optimum solution needs only 6 frequencies, three frequencies assigned to v_1 and v_3 , and three other frequencies to v_2 and v_4 . Thus in this case the ratio is $4/3$.

The above argument shows that the competitive ratio cannot be better than $4/3$, but it applies only to the absolute competitive ratio, because the optimal solution uses a constant number of frequencies in the above strategy. In fact, this bound can be improved to 1.5 if we use only one request per vertex instead of three. If we use k requests on each vertex instead, for some large k , this strategy will give a $4/3$ lower bound on the asymptotic ratio.

4. Competitive F-Systems

In this section we show that finding an R -competitive incremental algorithm for bipartite graphs can be reduced to an equivalent problem of constructing certain families of sets that we call F-systems.

Suppose that for any $c \in \{A, B\}$ and any integers t, k such that $0 < k \leq t$, we are given a set $F_{t,k}^c$ of positive integers (frequencies). Denote by $\mathcal{F} = \{F_{t,k}^c\}$ the family of those sets. Then \mathcal{F} is called an F -system if

$$(F1) |F_{t,k}^c| \geq k \text{ for all } c, t, k, \text{ and}$$

$$(F2) F_{t,k}^A \cap F_{t',k'}^B = \emptyset \text{ for all } k, k', t, t' \text{ such that } k + k' \leq \max(t, t').$$

An F-system is called R -competitive if for all t we have

$$\left| \bigcup_{k \leq \tau \leq t} (F_{\tau,k}^A \cup F_{\tau,k}^B) \right| \leq R \cdot t + \lambda, \quad (2)$$

where λ is a constant independent of t . The *competitive ratio* of \mathcal{F} is the smallest R for which \mathcal{F} is R -competitive.

Lemma 4.1. *For any $R \geq 1$, there is an R -competitive incremental algorithm for frequency allocation in bipartite graphs if and only if there is an R -competitive F-system.*

Proof. (\Rightarrow) Let \mathcal{A} be an R -competitive incremental algorithm. To prove this implication, we define a “universal” infinite bipartite graph $H = (A, B, E)$ and we will issue requests to this graph. For $c \in \{A, B\}$, the vertices in c have the form $(t, k)_c$, where $k \leq t$. Two vertices $(t, k)_A$ and $(t', k')_B$ are connected by an edge if $k + k' \leq \max(t, t')$.

The requests are issued in phases numbered $t = 1, 2, \dots$. In phase t , for each node $(t, k)_c$, we issue k requests to this node. Let $F_{t,k}^c$ be the set of frequencies that \mathcal{A} assigns to $(t, k)_c$. After phase t , by the definition of H and by (1), the optimum number of frequencies is t , so \mathcal{A} uses at most $Rt + \lambda$ frequencies, for some λ , implying (2) and proving that $\mathcal{F} = \{F_{t,k}^c\}$ is an R -competitive F-system.

(\Leftarrow) Let \mathcal{F} be an R -competitive F-system. We use \mathcal{F} to define an incremental algorithm \mathcal{A} that works as follows. Let $G = (A, B, E)$ be the given bipartite graph. Consider one step of the computation in which a new request arrives at a vertex $u \in c$, where $c \in \{A, B\}$. Denote by t the current optimum number of frequencies

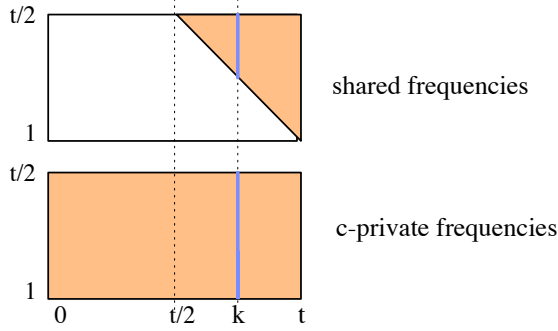


Figure 2: Frequency sets in the 1.5-competitive algorithm, represented by shaded regions. In this figure, we fix the value of t and show the frequency sets for $k \leq t$. The horizontal axis represents k and the vertical axis represents frequencies. $F_{t,k}^c$ is represented by a thick line segment on the vertical line corresponding to load k .

(including the one assigned to the new request), that is $t = \max_{\{v,w\} \in E} (\ell_v + \ell_w)$. Choose any frequency $f \in F_{t,k}^c$, for $k = \ell_u$, that is not yet used on u and assign f to this request. Such f exists, because by property (F1) we have $|F_{t,k}^c| \geq k$ and the number of frequencies assigned so far to u is $k - 1$.

Trivially, all frequencies assigned by \mathcal{A} to one node are different. We claim that adjacent nodes will be assigned different frequencies as well. Consider again a step where a frequency f is assigned to a k th request to a vertex u , when the optimum value is t , as described above. So $k = \ell_u$. Without loss of generality, assume $u \in A$. For an edge $\{u, v\} \in E$, let $k' = \ell_v$ be the current load at v . If g is a frequency assigned by \mathcal{A} to v , then, by the definition of \mathcal{A} , we have $g \in F_{t',k'}^B$ for some $t' \leq t$ and $k' \leq \min(t', k')$. Thus $k + k' \leq k + k' \leq t$, by the definition of t . Using (F2), we now get that $F_{t,k}^A \cap F_{t',k'}^B = \emptyset$, and therefore $f \neq g$.

Finally, when the optimum is t , then any frequency used by \mathcal{A} is from some set $F_{\tau,\kappa}^c$ for $\kappa \leq \tau \leq t$. Therefore \mathcal{A} is R -competitive, by the property (2) of \mathcal{F} . \square

5. An Upper Bound

We now design an R_0 -competitive incremental algorithm, for $R_0 = (18 - \sqrt{5})/11 \approx 1.433$. Using Lemma 4.1, it is sufficient to construct an R_0 -competitive F-system.

Intuitions. Our construction below may appear rather mysterious, so we begin by gradually introducing its main ideas. We will distinguish between two types of frequencies: private and shared. A-private frequencies will be used only in sets $F_{t,k}^A$, B-private frequencies will be used only in sets $F_{t,k}^B$, while shared frequencies can be used in some sets from both partitions A and B .

The competitive ratio of 2 can be easily achieved using only private frequencies. For each $c \in \{A, B\}$, let P^c be an infinite pool of c -private frequencies, with P^A and P^B disjoint. We simply let $F_{t,k}^c$ consist of the first k frequencies from P^c . Conditions (F1) and (F2) are trivially true. For any given t , the set on the left-hand side of (2) contains $2t$ frequencies, so (2) holds for $R = 2$, with $\lambda = 0$.

To improve the ratio to 1.5, we introduce an infinite pool S (disjoint with $P^A \cup P^B$) of shared frequencies. To avoid violations of (F2), we need to use these frequencies judiciously. Roughly, each $F_{t,k}^c$ will contain the first $t/2$ c -private frequencies, and if $k > t/2$ then it will also contain $k - t/2$ *last* shared frequencies numbered at most $t/2$, namely those from $t/2 - (k - t/2) + 1 = t - k + 1$ to $t/2$. (See Figure 2.) It is easy to verify that both (F1) and (2) are satisfied with $R = 1.5$. The intuition behind (F2) is this: $F_{t',k'}^B$ uses shared frequencies only when $k' > t'/2$. By symmetry, we can assume that $t' \leq t$. Then we have $k' \leq t - k$, which implies that $t'/2 \leq t - k$, so all shared frequencies in $F_{t',k'}^B$ are smaller than those in $F_{t,k}^A$.

To make the above idea more precise, for any real number $x \geq 0$ let

$$\begin{aligned} S_x &= \text{the first } \lfloor x \rfloor \text{ frequencies in } S, \\ P_x^c &= \text{the first } \lfloor x \rfloor \text{ frequencies in } P^c, \text{ for } c \in \{A, B\}. \end{aligned}$$

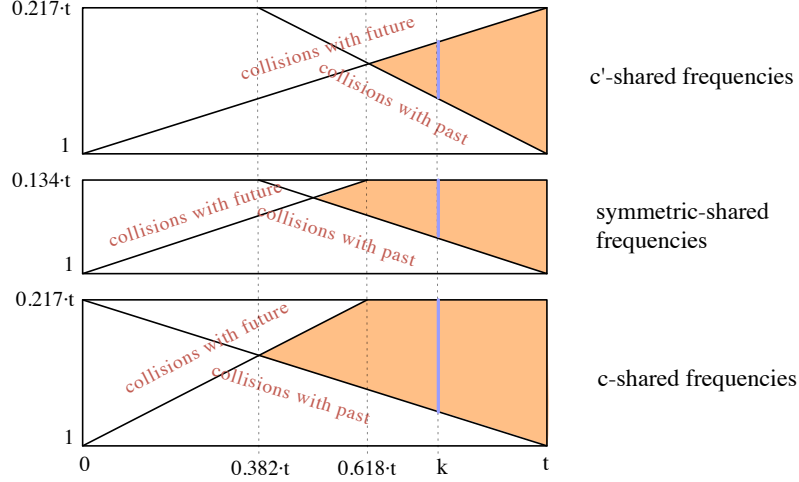


Figure 3: Frequency sets in the R_0 -competitive algorithm. We show only the shared frequencies, represented as in Figure 2.

We now let $\mathcal{F} = \{F_{t,k}^c\}$, where for $c \in \{A, B\}$ and $k \leq t$ we have

$$F_{t,k}^c = P_{t/2+1}^c \cup (S_{t/2} \setminus S_{t-k}).$$

We claim that \mathcal{F} is a 1.5-competitive F-system. We verify (F1) first. If $k \leq t/2$, then $|F_{t,k}^c| \geq |P_{t/2+1}^c| \geq k$. If $k > t/2$, then $t-k \leq t/2$, so $S_{t-k} \subseteq S_{t/2}$ and thus $|F_{t,k}^c| \geq \lfloor t/2 + 1 \rfloor + (\lfloor t/2 \rfloor - \lfloor t-k \rfloor) = k + 2\lfloor t/2 \rfloor + 1 - t \geq k$. So (F1) holds.

To verify (F2), pick any two pairs $k \leq t$ and $k' \leq t'$ with $k + k' \leq \max(t, t')$. Without loss of generality, assume $t' \leq t$. If $k' \leq t'/2$, then $t' - k' \geq t'/2$, so $F_{t',k'}^B \subseteq P^B$, and (F2) is trivial. If $k' > t'/2$, then $t'/2 \leq k' \leq t-k$, so $F_{t',k'}^B \subseteq P^B \cup S_{t'/2} \subseteq P^B \cup S_{t-k}$, which implies (F2) as well.

Finally, for $c \in \{A, B\}$ and $\kappa \leq \tau \leq t$, we have $F_{\tau,\kappa}^c \subseteq P_{t/2+1}^A \cup P_{t/2+1}^B \cup S_{t/2}$, so (2) holds with $R = 1.5$ and $\lambda = 2$, implying that \mathcal{F} is 1.5-competitive.

More intuition. It is useful to think of our constructions, informally, in terms of *collisions*. We designate some shared frequencies as forbidden in $F_{t,k}^c$, because they might be used in some sets $F_{t',k'}^c$ with $k + k' \leq \max(t, t')$. Depending on whether $t' \leq t$ or $t' > t$, we refer to those forbidden frequencies as “collisions with the past” and “collisions with the future”, respectively. Figure 2 illustrates this idea for our last construction. For $t' \leq t$ and $k' > t'/2$, each $F_{t',k'}^c$ uses shared frequencies numbered at most $t'/2 \leq k' \leq t-k$. (We ignore additive constants here.) Thus all shared frequencies that collide with the past are in the region below the line $f = t-k$, which complements the shaded region assigned to $F_{t,k}^c$.

An R_0 -competitive F-system. To achieve a ratio below 1.5 we need to use shared frequencies even when $k < t/2$. For such k near $t/2$, sets $F_{t,k}^A$ and $F_{t,k}^B$ will conflict and each will contain shared frequencies, so, unlike before, we need to assign their shared frequencies in a different order—in other words, we need to break symmetry. To achieve this, we introduce A-shared and B-shared frequencies. In sets $F_{t,k}^c$, for fixed c, t and increasing k , we first use c -private frequencies, then, starting at $k = t/\phi^2 \approx 0.382t$ we also use c -shared frequencies, then, starting at $k = t/2$ we add symmetric-shared frequencies, and finally, when k reaches $t/\phi \approx 0.618t$ we “borrow” c' -shared frequencies to include in this set. (See Figure 3.) We remark that symmetric-shared frequencies are still needed to reduce the ratio to $R_0 \approx 1.433$; with only private and c -shared frequencies we could only achieve ratio ≈ 1.447 .

Now, once the symmetry is broken, we can assign frequencies to $F_{t,k}^c$ more efficiently. The key to this is to consider “collisions with the future”, with sets $F_{t',k'}^c$ for $t' > t$.

Let’s take one more look at the construction for ratio 1.5. For $t' > t$, $F_{t,k}^c$ conflicts with $F_{t',k'}^c$ only if

$k + k' \leq t'$. The shared frequencies in this $F_{t',k'}^{c'}$ are numbered at least $t' - k' \geq k$, while $F_{t,k}^c$ uses shared frequencies only when $k \geq t/2$. So the frequencies representing collisions with the future for $F_{t,k}^c$ are outside the range $1, \dots, t/2$ of shared frequencies that can be used in $F_{t,k}^c$. (They are above the upper rectangle in Figure 2.) In this sense, this construction does not use collisions with the future.

Suppose that now we do include some collisions with the future in the range of shared frequencies. The crucial observation is this: if a frequency was used already in $F_{t'',k}^c$ for some $t'' < t$, then using it in $F_{t,k}^c$ cannot create any new collisions in the future. If the shared frequency sets for all t are represented by the same shape, this means that for every γ , either all points on the line $f = \gamma k$ create collisions in the future or none do. Thus the natural choice is to avoid frequencies in the the half-plane above the line $f = \gamma k$, for an appropriate γ . These collisions with the future, for a given $F_{t,k}^c$, are collisions with the past for $F_{t',k'}^{c'}$ with $t' > t$ and $k' \leq t' - k$. So, in the opposite partition, these collisions are represented by the half-plane below the line $f = \gamma(t' - k')$ (using the same γ). The optimization of the parameters for all three types of shared frequencies leads to our new algorithm.

The pools of c -shared and symmetric-shared frequencies are denoted S^c and Q , respectively. As before, for any real $x \geq 0$ we define

$$\begin{aligned} S_x^c &= \text{the first } \lfloor x \rfloor \text{ frequencies in } S^c, \text{ for } c \in \{A, B\}, \\ Q_x &= \text{the first } \lfloor x \rfloor \text{ frequencies in } Q. \end{aligned}$$

Let $\phi = (\sqrt{5} + 1)/2$ be the golden ratio. We have $R_0 = (\phi + 5)/(\phi + 3)$. Our construction uses three other constants

$$\alpha = \frac{2}{\phi + 3} \approx 0.433, \quad \beta = \frac{1}{\phi + 3} \approx 0.217, \quad \text{and} \quad \rho = \frac{\phi - 1}{\phi + 3} \approx 0.134.$$

We have the following useful identities: $\alpha = R_0 - 1$, $\beta = \alpha/2$, $\rho = \beta/\phi$, and $2\alpha + 2\beta + \rho = R_0$.

We define $\mathcal{F} = \{F_{t,k}^c\}$, where for any $t \geq k \geq 0$ we let

$$\begin{aligned} F_{t,k}^c &= P_{\alpha t + 4}^c \cup (S_{\beta \cdot \min(t, \phi k)}^c \setminus S_{\beta(t-k)}^c) \cup (S_{\beta k}^{c'} \setminus S_{\phi \beta(t-k)}^{c'}) \\ &\quad \cup (Q_{\rho \cdot \min(t, \phi k)} \setminus Q_{\phi \rho(t-k)}). \end{aligned} \tag{3}$$

We now show that \mathcal{F} is an R_0 -competitive F-system. We start with (2). For $\kappa \leq \tau \leq t$ and $c \in \{A, B\}$ we have

$$\begin{aligned} F_{\tau, \kappa}^c &\subseteq P_{\alpha \tau + 4}^c \cup S_{\beta \tau}^c \cup S_{\beta \kappa}^{c'} \cup Q_{\rho \tau} \subseteq P_{\alpha t + 4}^c \cup S_{\beta t}^c \cup S_{\beta t}^{c'} \cup Q_{\rho t} \\ &\subseteq P_{\alpha t + 4}^A \cup P_{\alpha t + 4}^B \cup S_{\beta t}^A \cup S_{\beta t}^B \cup Q_{\rho t}. \end{aligned}$$

This last set has cardinality at most $(2\alpha + 2\beta + \rho)t + 8 = R_0 t + 8$, so (2) holds.

Next, we show (F2). By symmetry, we can assume that $t' \leq t$ in (F2), so $k' \leq t - k$. Then

$$F_{t',k'}^B \subseteq P^B \cup S_{\phi \beta k'}^B \cup S_{\beta k'}^A \cup Q_{\phi \rho k'} \subseteq P^B \cup S_{\phi \beta(t-k)}^B \cup S_{\beta(t-k)}^A \cup Q_{\phi \rho(t-k)},$$

and this set is disjoint with $F_{t,k}^A$ by (3). Thus $F_{t,k}^A \cap F_{t',k'}^B = \emptyset$, as needed.

Finally, we prove (F1), namely that $|F_{t,k}^c| \geq k$. We distinguish two cases.

Case 1: $k > t/\phi$. This implies that $\min(t, \phi k) = t$, so in (3) we have $S_{\beta \cdot \min(t, \phi k)}^c = S_{\beta t}^c$ and $Q_{\rho \cdot \min(t, \phi k)} = Q_{\rho t}$. Thus

$$\begin{aligned} |F_{t,k}^c| &\geq [\alpha t + 3] + [\beta t - \beta(t - k) - 1] + [\beta k - \phi \beta(t - k) - 1] + [\rho t - \phi \rho(t - k) - 1] \\ &= (\alpha - \phi \beta - (\phi - 1)\rho)t + (2\beta + \phi \beta + \phi \rho)k = k, \end{aligned}$$

using the substitutions $\alpha = 2\beta$ and $\rho = \beta/\phi$. Note that this case is asymptotically tight as the algorithm uses all three types of shared frequencies (and the corresponding terms are non-negative).

Case 2: $k \leq t/\phi$. The case condition implies that $\phi k \leq t$, so $S_{\beta \cdot \min(t, \phi k)}^c = S_{\phi \beta k}^c$, $Q_{\rho \cdot \min(t, \phi k)} = Q_{\phi \rho k}$, and $S_{\beta k}^{c'} \setminus S_{\phi \beta(t-k)}^{c'} = \emptyset$. Therefore

$$\begin{aligned} |F_{t,k}^c| &\geq [\alpha t + 3] + [\phi \beta k - \beta(t-k) - 1] + [\phi \rho k - \phi \rho(t-k) - 1] \\ &= (\alpha - \beta - \phi \rho)t + ((\phi + 1)\beta + 2\phi \rho)k + 1 = k + 1, \end{aligned}$$

using $\alpha = 2\beta$ and $\rho = \beta/\phi$ again. Note that this case is (asymptotically) tight only for $k > t/2$ when c -shared and symmetric-shared frequencies are used. For $k \leq t/2$, the term corresponding to symmetric-shared frequencies is negative.

Summarizing, we conclude that \mathcal{F} is indeed an R_0 -competitive F-system. Therefore, using Lemma 4.1, we get our upper bound:

Theorem 5.1. *There is an R_0 -competitive incremental algorithm for frequency allocation on bipartite graphs, where $R_0 = (18 - \sqrt{5})/11 \approx 1.433$.*

6. A Lower Bound

In this section we show that if $R < 10/7$, then there is no R -competitive incremental algorithm for frequency allocation in bipartite graphs. By Lemma 4.1, it is sufficient to show that there is no R -competitive F-system.

The general intuition behind the proof is that we try to reason about the sets $Z_t = F_{t,\gamma t}^A \cap F_{t,\gamma t}^B$ for a suitable constant γ . These sets should correspond to the symmetric-shared frequencies from our algorithm, for γ such that no c' -shared frequencies are used in the c -partition. If Z_t is too small, then both partitions use mostly different frequencies and this yields a lower bound on the competitive ratio. If Z_t is too large, then for a larger t and suitable k , the frequencies cannot be used for either partition, and hopefully this allows to improve the lower bound. We are not able to do exactly this. Instead, for a variant of Z_t , we show a recurrence essentially saying that if the set is too large, then for some larger t , it must be proportionally even larger, leading to a contradiction.

We now proceed with the proof. For $c \in \{A, B\}$, let $F_t^c = \bigcup_{\kappa \leq t \leq \kappa} F_{\tau, \kappa}^c$. Towards contradiction, suppose that an F-system \mathcal{F} is R -competitive for some $R < 10/7$. Then \mathcal{F} satisfies the definition of competitiveness (2) for some positive integer λ , that is $|F_t^A \cup F_t^B| \leq Rt + \lambda$. Choose a sufficiently large integer θ for which $R < 10/7 - 1/\theta$.

We first identify shared frequencies in \mathcal{F} . The set of *level- t shared* frequencies is defined as $S_t = F_t^A \cap F_t^B$.

Lemma 6.1. *For any t , we have $|S_t| \geq (2 - R)t - \lambda$.*

Proof. This is quite straightforward. By (F1) we have $|F_t^c| \geq t$ for each c , so $|S_t| = |F_t^A| + |F_t^B| - |F_t^A \cup F_t^B| \geq 2t - (Rt + \lambda) = (2 - R)t - \lambda$. \square

Now, let $S_{2t,t} = S_{2t} \cap (F_{2t,t}^A \cup F_{2t,t}^B)$ be the level- $2t$ shared frequencies that are used in $F_{2t,t}^A$ or $F_{2t,t}^B$. Each such frequency can only be in one of these sets because $F_{2t,t}^A \cap F_{2t,t}^B = \emptyset$.

Lemma 6.2. *For any t , we have $|S_{2t,t}| \geq (6 - 4R)t - 2\lambda$.*

Proof. By definition, $F_{2t,t}^A \cup F_{2t,t}^B \cup S_{2t} \subseteq F_{2t}^A \cup F_{2t}^B$, and thus (2) implies

$$\begin{aligned} 2Rt + \lambda &\geq |F_{2t,t}^A \cup F_{2t,t}^B \cup S_{2t}| \\ &= |F_{2t,t}^A \cup F_{2t,t}^B| + |S_{2t}| - |(F_{2t,t}^A \cup F_{2t,t}^B) \cap S_{2t}| \\ &= |F_{2t,t}^A| + |F_{2t,t}^B| + |S_{2t}| - |S_{2t,t}|, \end{aligned}$$

where the equations follow from the inclusion-exclusion principle, disjointness of $F_{2t,t}^A$ and $F_{2t,t}^B$, and the definition of $S_{2t,t}$. Transforming this inequality, we get

$$|S_{2t,t}| \geq |F_{2t,t}^A| + |F_{2t,t}^B| + |S_{2t}| - (2Rt + \lambda) \geq (6 - 4R)t - 2\lambda,$$

as claimed, by property (F1) and Lemma 6.1. \square

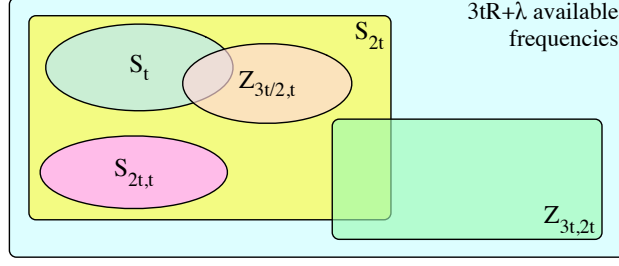


Figure 4: Illustration of Lemma 6.3.

For any even t define $Z_{3t/2,t} = F_{3t/2,t}^A \cap F_{3t/2,t}^B$. In the rest of the lower-bound proof we will set up a recurrence relation for the cardinality of sets $S_t \cup Z_{3t/2,t}$. The next step is the following lemma.

Lemma 6.3. *For any even t , we have $|S_{2t} \setminus Z_{3t,2t}| \geq |S_t \cup Z_{3t/2,t}| + |S_{2t,t}|$.*

Proof. From the definition, the sets $S_t \cup Z_{3t/2,t}$ and $S_{2t,t}$ are disjoint and contained in $S_{2t} - Z_{3t,2t}$ (see Figure 4.) This immediately implies the lemma. \square

Lemma 6.4. *For any even t , we have $|Z_{3t,2t}| \geq |S_t \cup Z_{3t/2,t}| - (3R - 4)t - \lambda$.*

Proof. As $F_{3t,2t}^A \cup F_{3t,2t}^B \cup S_t \cup Z_{3t/2,t} \subseteq F_{3t}^A \cup F_{3t}^B$, inequality (2) implies

$$\begin{aligned}
3Rt + \lambda &\geq |F_{3t,2t}^A \cup F_{3t,2t}^B \cup S_t \cup Z_{3t/2,t}| \\
&= |F_{3t,2t}^A \cup F_{3t,2t}^B| + |S_t \cup Z_{3t/2,t}| \\
&= |F_{3t,2t}^A| + |F_{3t,2t}^B| - |F_{3t,2t}^A \cap F_{3t,2t}^B| + |S_t \cup Z_{3t/2,t}| \\
&= |F_{3t,2t}^A| + |F_{3t,2t}^B| - |Z_{3t,2t}| + |S_t \cup Z_{3t/2,t}|,
\end{aligned}$$

where the identities follow from the inclusion-exclusion principle, the fact that $F_{3t,2t}^A \cup F_{3t,2t}^B$ and $S_t \cup Z_{3t/2,t}$ are disjoint, and the definition of $Z_{3t,2t}$. Transforming this inequality, we get

$$\begin{aligned}
|Z_{3t,2t}| &\geq |F_{3t,2t}^A| + |F_{3t,2t}^B| + |S_t \cup Z_{3t/2,t}| - (3Rt + \lambda) \\
&\geq |S_t \cup Z_{3t/2,t}| - (3R - 4)t - \lambda,
\end{aligned}$$

as claimed, by property (F1). \square

We are now ready to derive our recurrence. By adding the inequalities in Lemma 6.3 and Lemma 6.4, taking into account that $|S_{2t} \setminus Z_{3t,2t}| + |Z_{3t,2t}| = |S_{2t} \cup Z_{3t,2t}|$, and then applying Lemma 6.2, for any even t we get

$$\begin{aligned}
|S_{2t} \cup Z_{3t,2t}| &\geq 2 \cdot |S_t \cup Z_{3t/2,t}| + |S_{2t,t}| - (3R - 4)t - \lambda \\
&\geq 2 \cdot |S_t \cup Z_{3t/2,t}| + (10 - 7R)t - 3\lambda.
\end{aligned} \tag{4}$$

For $i = 0, 1, \dots, \theta$, define $t_i = 6\theta\lambda 2^i$ and $\gamma_i = |S_{t_i} \cup Z_{3t_i/2,t_i}|/t_i$. (Note that each t_i is even.) Since $S_{t_i} \cup Z_{3t_i/2,t_i} \subseteq S_{2t_i}$, we have that $\gamma_i \leq |S_{2t_i}|/t_i \leq 2R + 1/(6\theta) < 3$. Dividing recurrence (4) with $t = t_i$ by $t_{i+1} = 2t_i$, we obtain

$$\gamma_{i+1} \geq \gamma_i + 5 - 7R/2 - 3\lambda/(2t_i) \geq \gamma_i + 7/(2\theta) - 1/(4\theta) \geq \gamma_i + 3/\theta,$$

for $i = 0, 1, \dots, \theta - 1$. But then we have $\gamma_\theta \geq \gamma_0 + 3 \geq 3$, which contradicts our earlier bound $\gamma_i < 3$, completing the proof. Thus we have proved the following.

Theorem 6.5. *If \mathcal{A} is an R -competitive incremental algorithm for frequency allocation on bipartite graphs, then $R \geq 10/7 \approx 1.428$.*

As a final remark we observe that our lower bound works even if the additive constant λ is allowed to depend on the actual graph. More specifically, for every $R < 10/7$ we can construct a single finite graph G so that no online algorithm is R -competitive on this graph. In our lower bound argument above, we can restrict our attention to sets F_{t_i, t_i}^c , F_{2t_i, t_i}^c and $F_{3t_i/2, t_i}^c$, for $i = 0, 1, \dots, \theta$ and $c = A, B$. To construct G , we follow the construction of the universal graph in the the proof of the “ \Rightarrow ” implication of Lemma 4.1, except that we only use the nodes representing the sets from our lower bound proof, namely nodes $(t_i, t_i)_c$, $(2t_i, t_i)_c$, and $(3t_i/2, t_i)_c$. For a fixed θ , the graphs obtained for different values of λ are isomorphic, as all the indices t_i scale linearly with λ . In other words, only the loads on the vertices depend on λ , not the underlying graph. So, instead of using different isomorphic graphs, we can use different sequences corresponding to different values of λ on a single graph G .

7. Final Comments

We proved that the competitive ratio for incremental frequency allocation on bipartite graphs is between 1.428 and 1.433, improving the previous bounds of 1.33 and 1.5. Closing the remaining gap, small as it is, remains an intriguing open problem. Besides completing the analysis of this case, the solution is likely to give new insights into the general problem.

Two other obvious directions of study are to prove better bounds for the dynamic case and for k -partite graphs. Our idea of distinguishing “collisions with the past” and “collisions with the future” should be useful to derive upper bounds for these problems. The concept of F-systems extends naturally to k -partite graphs, but with a caveat: For $k \geq 3$ the maximum load on a k -clique is only a lower bound on the optimum (unlike for $k = 2$, where the equality holds). Therefore in Lemma 4.1 only one direction holds. This lemma is still sufficient to establish upper bounds on the competitive ratio, and it is possible that a lower bound can be proved using graphs where the optimum number of frequencies is equal to the maximum load of a k -clique.

Acknowledgements

Marek Chrobak was supported by NSF Grant CCF-0729071. Łukasz Jeż was partially supported by MNiSW grant N N206 368839, 2010–2013, a scholarship co-financed by an ESF project *Human Capital*, grant IAA100190902 of GA AV ČR, and GRaDR EUROGIGA GAČR GIG/11/E023 project. Jiří Sgall was partially supported by Inst. for Theor. Comp. Sci., Prague (project P202/12/G061 of GA ČR).

References

- [1] C. McDiarmid, B. Reed, Channel assignment and weighted colouring, *Networks* 36 (2000) 114–117.
- [2] R. A. Murphey, P. M. Pardalos, Mauricio, M. G. Resende, Frequency assignment problems, in: *Handbook of Combinatorial Optimization*, Kluwer Academic Publishers, 1999, pp. 295–377.
- [3] K. Aardal, S. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino, A. Sassano, Models and solution techniques for frequency assignment problems, *4OR: Quarterly Journal of the Belgian, French and Italian Operations Research Societies* 1 (2003) 261–317.
- [4] L. Narayanan, S. M. Shende, Static frequency assignment in cellular networks, *Algorithmica* 29 (2001) 396–409.
- [5] J. W.-T. Chan, F. Y. L. Chin, D. Ye, Y. Zhang, Online frequency allocation in cellular networks, in: *Proc. 19th Symp. on Parallel Algorithms and Architectures (SPAA)*, 2007, pp. 241–249.
- [6] J. W.-T. Chan, F. Y. L. Chin, D. Ye, Y. Zhang, Absolute and asymptotic bounds for online frequency allocation in cellular networks, *Algorithmica* 58 (2010) 498–515.
- [7] J. W.-T. Chan, F. Y. L. Chin, D. Ye, Y. Zhang, H. Zhu, Frequency allocation problem for linear cellular networks, in: *Proc. 17th International Symp. on Algorithms and Computation (ISAAC)*, 2006, pp. 61–70.
- [8] M. Chrobak, J. Sgall, Three results on frequency assignment in linear cellular networks, *Theoretical Computer Science* 411 (2010) 131–137.
- [9] F. Y. L. Chin, Y. Zhang, H. Zhu, A 1-local 13/9-competitive algorithm for multicoloring hexagonal graphs, in: *Proc. 13th Annual International Computing and Combinatorics Conf. (COCOON)*, 2007, pp. 526–536.