Optimal analysis of Best Fit bin packing

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Abstract. In early seventies it was shown that the *asymptotic* approximation ratio of BestFit bin packing is equal to 1.7. We prove that also the *absolute* approximation ratio for BestFit bin packing is exactly 1.7, improving the previous bound of 1.75. This means that if the optimum needs Opt bins, BestFit always uses at most $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins. Furthermore we show *matching lower bounds* for all values of Opt, i.e., we give instances on which BestFit uses exactly $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins. Thus we completely settle the worst-case complexity of BestFit bin packing after more than 40 years of its study.

1 Introduction

Bin packing is a classical combinatorial optimization problem in which we are given an instance consisting of a sequence of items with rational sizes between 0 and 1, and the goal is to pack these items into the smallest possible number of bins of unit size. BestFit algorithm packs each item into the most full bin where it fits, possibly opening a new bin if the item does not fit into any currently open bin. A closely related FirstFit algorithm packs each item into the first bin where it fits, again opening a new bin only if the item does not fit into any currently open bin.

Johnson's thesis [8] on bin packing together with Graham's work on scheduling [6, 7] belong to the early influential works that started and formed the whole area of approximation algorithms. The proof that the asymptotic approximation ratio of FirstFit and BestFit bin packing is 1.7 given by Ullman [14] and subsequent works by Garey et al. and Johnson et al. [5, 10] were among these first results on approximation algorithms.

We prove that also the *absolute* approximation ratio for BestFit bin packing is exactly 1.7, i.e., if the optimum needs Opt bins, BestFit always uses at most $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. This builds upon and substantially generalizes our previous upper bound for FirstFit from [3]. For the comparison of the techniques, see the beginning of Section 4. Furthermore we show *matching lower bounds* for *all* values of Opt, i.e., we give instances on which BestFit and FirstFit use exactly $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. This is also the first construction of an instance that has absolute approximation ratio exactly 1.7 for an arbitrarily large Opt.

Note that the upper bound for BestFit is indeed a generalization of the bound for FirstFit: The items in any instance can be reordered so that they arrive in the order of bins in the FirstFit packing. This changes neither FirstFit, nor the optimal packing. Thus it is sufficient to analyze FirstFit on such instances. On the other hand, on them BestFit behaves exactly as FirstFit, as there is always a single bin where the new item fits. Thus any lower bound for FirstFit applies immediately to BestFit and any upper bound for FirstFit is equivalent to a bound for BestFit for this very restricted subset of instances. To demonstrate that the extension of the absolute bound from FirstFit to BestFit is by far not automatic, we present a class of any-fit-type algorithms for which the asymptotic bound of 1.7 holds, but the absolute bound does not.

History and related work. The upper bound on BestFit (and FirstFit) was first shown by Ullman in 1971 [14]; he proved that for any instance, $\mathrm{BF}, \mathrm{FF} \leq 1.7 \cdot \mathrm{OPT} + 3$, where BF, FF and Opt denote the number of bins used by BestFit, FirstFit and the optimum, respectively. Still in seventies, the additive term was improved first in [5] to 2 and then in [4] to $\mathrm{BF} \leq \lceil 1.7 \cdot \mathrm{OPT} \rceil$; due to integrality of BF and Opt this is equivalent to $\mathrm{BF} \leq 1.7 \cdot \mathrm{OPT} + 0.9$. Recently the additive term of the asymptotic bound was improved for FirstFit to $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT} + 0.7$ in [16] and to $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$ in [3].

The absolute approximation ratio of FirstFit and BestFit was bounded by 1.75 by Simchi-Levy [13]. Recent improvements again apply only to FirstFit: after bounds of $12/7 \approx 1.7143$ by Xia and Tan [16] and Boyar et al. [1] and $101/59 \approx 1.7119$ by Németh [11], the tight bound of $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$ was given in our previous work [3].

For the lower bound, the early works give examples both for the asymptotic and absolute ratios. The example for the asymptotic bound gives $\mathrm{FF} = 17k$ whenever $\mathrm{OPT} = 10k + 1$, thus it shows that the asymptotic upper bound of 1.7 is tight, see [14, 5, 10]. For the absolute ratio, an example is given with $\mathrm{FF} = 17$ and $\mathrm{OPT} = 10$, i.e., an instance with approximation ratio exactly 1.7 [5, 10], but no such example was known for large Opt. In our previous work [3] we have given lower bound instances with $\mathrm{BF} = \mathrm{FF} = \lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ whenever $\mathrm{OPT} \not\equiv 0, 3 \pmod{10}$.

We have mentioned only directly relevant previous work. Of course, there is much more work on bin packing, in particular there exist asymptotic approximation schemes for this problem, as well as many other algorithms. We refer to the survey [2] or to the recent excellent book [15].

Organization of the paper. The crucial technique of the upper bound is a combination of amortization and weight function analysis, following the scheme of our previous work [12, 3]. We present it first in Section 2 to give a simple proof of the asymptotic bound for BestFit and any-fit-type algorithms. We prove the lower bound in Section 3, as it illustrates well the issues that we need to deal with in the upper bound proof, which is then given in Section 4. Most proofs are omitted, but we try to explain the main ideas behind them. For a version with all proofs, see `http://iuuk.mff.cuni.cz/~sgall/ps/BF.pdf`.

2 Notations and the simplified asymptotic bound

Let us fix an instance I with items $a_1, \ldots, a_n$ and denote the number of bins in the BestFit and optimal solutions by BF and Opt, respectively. We will often identify an item and its size. For a set of items A, let $s(A) = \sum_{a \in A} a$, i.e., the total size of items in A and also for a set of bins $\mathcal{A}$, let $s(\mathcal{A}) = \sum_{A \in \mathcal{A}} s(A)$. Furthermore, let $S = s(I)$ be the total size of all items of I. Obviously $S \leq \text{Opt}$.

We classify the items by their sizes: items $a \leq 1/6$ are **small**, items $a \in (1/6, 1/2]$ are **medium**, and items $a > 1/2$ are **huge**. A bin is called a **k-bin** or **k^+-bin**, if it contains exactly k items or at least k items, respectively, in the final packing. Furthermore, the **rank** of a bin is the number of medium and huge items in it. An item is called **k-item** if BF packs it into a k-bin.

The bins in the BF packing are ordered by the time they are opened (i.e., when the first item is packed into them). Expressions like "before", "after", "first bin", "last bin" refer to this ordering. At any time during the packing, the **level of a bin** is the total size of items currently packed in it, while by **size of a bin** we always mean its final level. A **level of an item** a denotes the level of the bin where a is packed, just before the packing of a.

The following properties of BestFit follow easily from its definition.

Lemma 2.1. *At any moment, in the* BF *packing the following holds:*
(i) *The sum of levels of any two bins is greater than 1. In particular, there is at most one bin with level at most* $1/2$.
(ii) *Any item a with level at most* $1/2$ *(i.e., packed into the single bin with level at most* $1/2$*) does not fit into any bin open at the time of its arrival, except for the bin where the item a is packed.*
(iii) *If there are two bins B, B' with level at most* $2/3$*, in this order, then either B' contains a single item or the first item in B' is huge.* ☐

To illustrate our technique, we now present a short proof of the asymptotic ratio 1.7 for BestFit. It uses the same weight function as the traditional analysis of BestFit. (In some variants the weight of an item is capped to be at most 1, which makes almost no difference in the analysis.) To use amortization, we split the weight of each item a into two parts, namely its bonus $\overline{w}(a)$ and its scaled size $\overline{\overline{w}}(a)$, defined as

$$\overline{w}(a) = \begin{cases} 0 & \text{if } a \leq \frac{1}{6}\,, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } a \in \left(\frac{1}{6}, \frac{1}{3}\right)\,, \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right]\,, \\ 0.4 & \text{if } a > \frac{1}{2}\,. \end{cases}$$

For every item a we define $\overline{\overline{w}}(a) = \frac{6}{5}a$ and its weight is $w(a) = \overline{\overline{w}}(a) + \overline{w}(a)$. For a set of items B, $w(B) = \sum_{a \in B} w(a)$ denotes the total weight, similarly for $\overline{w}$ and $\overline{\overline{w}}$.

It is easy to observe that the weight of any bin B, i.e., of any set with $s(B) \leq 1$, is at most 1.7: The scaled size of B is at most 1.2, so we only need to check that $\overline{w}(B) \leq 0.5$. If B contains no huge item, there are at most 5 items with non-zero $\overline{w}(a)$ and $\overline{w}(a) \leq 0.1$ for each of them. Otherwise the huge item has bonus 0.4; there are at most two other medium items with non-zero bonus and it is easy to check that their total bonus is at most 0.1. This implies that the weight of the whole instance is at most $1.7 \cdot \text{OPT}$.

The key part is to show that, on average, the weight of each BF-bin is at least 1. Lemma 2.2 together with Lemma 2.1 implies that for almost all bins with two or more items, its scaled size plus the bonus of the *following* such bin is at least 1.

Lemma 2.2. *Let B be a bin such that $s(B) \geq 2/3$ and let c, c' be two items that do not fit into B, i.e., $c, c' > 1 - s(B)$. Then $\overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') \geq 1$.*

Proof. If $s(B) \geq 5/6$, then $\overline{\overline{w}}(B) \geq 1$ and we are done. Otherwise let $x = 5/6 - s(B)$. We have $0 < x \leq 1/6$ and thus $c, c' > 1/6 + x$ implies $\overline{w}(c), \overline{w}(c') > \frac{3}{5}x$. We get $\overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. □

Any BF-bin D with a huge item has $\overline{w}(D) \geq 0.4$ and $\frac{6}{5}s(D) > 0.6$, thus $w(D) > 1$.

For the amortization, consider all BF-bins B with two or more items, size $s(B) \geq 2/3$, and no huge item. For any such bin except for the last one choose C as the next bin with the same properties. Since C has no huge item, its first two items c, c' have level at most 1/2 and by Lemma 2.1(ii) they do not fit into B. Lemma 2.2 implies $\overline{\overline{w}}(B) + \overline{w}(C) \geq \overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') \geq 1$.

Summing all these inequalities (note that each bin is used at most once as B and at most once as C) and $w(D) > 1$ for the bins with huge items we get $w(I) \geq \text{BF} - 3$. The additive constant 3 comes from the fact that we are missing an inequality for at most three BF-bins: the last one from the amortization sequence, possibly one bin B with two or more items and $s(B) < 2/3$ (cf. Lemma 2.1(iii)) and possibly one bin B with a single item and $s(B) < 1/2$ (cf. Lemma 2.1(i)). Combining this with the previous bound on the total weight, we obtain $\text{BF} - 3 \leq w(I) \leq 1.7 \cdot \text{OPT}$ and the asymptotic bound follows.

This simple proof holds for a wide class of any-fit-type algorithms: Call an algorithm a RAAF (relaxed almost any fit) algorithm, if it uses the bin with level at most 1/2 only when the item does not fit into any previous bin (Lemma 2.1(i) implies that there is always at most one such bin). Our proof of the asymptotic ratio can be tightened so that the additive constant is smaller:

Theorem 2.3. *For any RAAF algorithm A and any instance of bin packing we have $A \leq \lfloor 1.7 \cdot \text{OPT} + 0.7 \rfloor \leq \lceil 1.7 \cdot \text{OPT} \rceil$.* □

The proof is given in the full version, where we also give an example of a RAAF algorithm which does not satisfy the absolute bound of 1.7. The asymptotic bound for almost any fit (AAF) algorithms was proved in [8, 9], where the original AAF condition prohibits packing an item in the smallest bin if that bin is unique and the item does fit in some previous bin (but the restriction holds also if the smallest bin is larger than 1/2). Theorem 2.3 improves the additive term and generalizes the bound from AAF to the slightly less restrictive RAAF condition (although it seems that the original proof also uses only the RAAF condition).

3 Lower bound

The high level scheme of the lower bound for $\textsc{Opt} = 10k$ is this: For a tiny $\varepsilon > 0$, the instance consists of $\textsc{Opt}$ items of size approximately $1/6$, followed by $\textsc{Opt}$ items of size approximately $1/3$, followed by $\textsc{Opt}$ items of size $1/2+\varepsilon$. The optimum packs in each bin one item from each group. BestFit packs the items of size about $1/6$ into $2k$ bins with 5 items, with the exception of the first and last of these bins that will have 6 and 4 items, respectively. The items of size about $1/3$ are packed in pairs. To guarantee this packing, the sizes of items differ from $1/6$ and $1/3$ in both directions by a small amount δ_i which is exponentially decreasing, but greater than ε for all i. This guarantees that only the item with the largest δ_i in a bin is relevant for its final size and this in turn enables us to order the items so that no additional later item fits into these bins.

Theorem 3.1. *For all values of* $\textsc{Opt}$*, there exists an instance* I *with* $\mathrm{FF} = \mathrm{BF} = \lfloor 1.7 \cdot \textsc{Opt} \rfloor$. □

4 Upper bound

At the high level, we follow the weight function argument from the simple proof in Section 2. As we have seen, the BF packing in the lower bound contains three types of bins that play different roles. To obtain the tight upper bound, we analyze them separately. For two of these types we can argue easily that the weight of each bin is at least 1: First, the bins with size at least $5/6$, called big bins below; these are the initial bins in the lower bound containing the items of size approximately $1/6$. Second, the 1-bins, called dedicated bins; these are the last bins with items $1/2+\varepsilon$ in the lower bound. The remaining bins, common bins, are the middle bins of size approximately $2/3$ with items of size around $1/3$ (except for the first bin) in the lower bound. They are analyzed using the amortization lemma. This general scheme has several obstacles which we describe now, together with the intuition behind their solution.

Obstacle 1: There can be one dedicated bin with item $d_0 < 1/2$. We need to change its bonus to approximately 0.4, to guarantee a sufficient weight of this bin. This in turn possibly forces us to decrease the bonus of one huge item f_1 to 0.1, if d_0 and f_1 are in the same $\textsc{Opt}$-bin, so that the $\textsc{Opt}$-bin has weight at most 1.7.

Obstacle 2: The amortization lemma needs two items that do not fit into the previous bin. Unlike FirstFit, BestFit does not guarantee this, if the first item in a bin is huge. If this first item is not f_1, we can handle such bins, called huge-first bins, similarly as dedicated bins. If this happens for f_1, we need to argue quite carefully to find the additional bonus. This case, called the freaky case, is the most complicated part of our analysis.

Obstacle 3: Even on the instances similar to our lower bound, the amortization leaves us with the additive term of 0.1, because we cannot use the amortization on the last common bin, and its scaled weight is only about 0.8 if its size is

around 2/3. Here the parity of the items of size around 1/3 comes into play: Typically they come in pairs in BF-bins, as in the lower bound, but for odd values of Opt one of them is missing or is in a FirstFit bin of 3 or more items. This allows us to remove the last 0.1 of the additive term, using the mechanism of an exceptional set, see Definition 4.9.
Obstacle 4: If the last common bin is smaller than 2/3, the problem with amortizing it is even larger. Fortunately, then the previous common bins are larger than 2/3 and have additional weight that can compensate for this, using a rather delicate argument, see Proposition 4.11.

Notations and preliminary lemmata

We classify the BF bins into four groups.

Any 1-bin D is a **dedicated bin**; $\mathcal{D}$ denotes the set of all dedicated bins and δ their number. If some dedicated bin has size at most 1/2, denote the item in it d_0 and let $\Delta = 1/2 - d_0$; otherwise d_0 and Δ are undefined. Lemma 2.1(i) implies that there is at most one such item; also we shall see that in the tightest case Δ is close to 0.

If d_0 is defined, there may exist a (unique) huge item in its Opt-bin. In that case, denote it f_1 for the rest of the proof and leave f_1 undefined otherwise. Furthermore, if f_1 is the first item in a BF-bin, denote that bin F for the rest of the proof; otherwise let F undefined. Note that F cannot be a 1-bin as otherwise d_0 would fit there contradicting Lemma 2.1(i). Let f_2 be the second item in F.

If the first item of a 2^+-bin H is huge and $H \neq F$, we call H a **huge-first** bin; $\mathcal{H}$ denotes the set of all huge-first bins and η their number.

If a 2^+-bin B satisfies $s(B) \geq 5/6$ and it is not in $\mathcal{H}$, we call it a **big bin**; $\mathcal{B}$ denotes the set of all big bins and β their number.

Any remaining bin (i.e., any 2^+-bin with size less than 5/6 and the first item small or medium, and also F if it is defined and not a big bin) is called a **common bin**; $\mathcal{C}$ denotes the set of all common bins, and γ their number.

An item is called an **H-item**, if it is d_o or a huge item different from f_1 (if defined). Note that each Opt-bin and each BF-bin contains at most one H-item.

The definitions imply that in every big or common bin different from F (if defined), the first item is small or medium. Then Lemma 2.1(ii) implies that the first two items of the bin do not fit into any previous bin.

Throughout the proof we distinguish two cases depending on the bin F.

If F is not defined, or it is a big bin, or f_2 does not fit into any previous common bin, then we call this the **regular case** and all the common bins, including F if it is defined and a common bin, are called **regular bins**.

If F is defined and it is a common bin, and f_2 would fit into some previous common bin at the time of its packing, fix one such bin G for the rest of the proof. We call this the **freaky case** and F the **freaky bin**. All the other common bins are called **regular bins**.

In both cases, denote the set of all regular bins by $\mathcal{R}$ and their number by ρ, furthermore number the regular bins $C_1, \ldots, C_\rho$, ordered by the time of their opening. In the freaky case, let g be the index of bin G in this ordering, i.e., let $C_g = G$. Note that $\rho = \gamma$ in the regular case and $\rho = \gamma - 1$ in the freaky case.

In the following lemma we significantly reduce the set of instances that we need to consider in our proof. Our goal is to reorder or remove the items in the sequence so that BestFit packs most items similarly as FirstFit. For these transformations, we use two important properties of BestFit that follow from its definition. First, if we remove all the items from a BF-bin from the instance, the packing of the remaining items into the remaining bins does not change; often we use this so that we move the items to a later position in the instance and then this implies that the packing of the initial segment before the new position of the moved items does not change. Second, if two instances lead to the same configuration and we extend them by the same set of items, then the resulting configurations are also the same, where the configuration is the current multiset of levels of BF-bins. (This does not hold for FirstFit, as permuting the bins can change the subsequent packing, but the configuration is the same.)

Lemma 4.1. *If there exists an instance with* BF $> 1.7 \cdot$ Opt*, then there exists such an instance I that in addition satisfies the following properties:*

(i) *All the* 1*-items form a final segment of the input instance.*

(ii) *If a* BF *bin B contains an item a such that for all other* BF *bins B' we have $a + s(B') > 1$ then B is an* 1*-bin.*

(iii) *In each* BF 2^+*-bin, the first two items are a_{j-1} and a_j for some j (i.e., they are adjacent in I). Furthermore, these two items are packed into different bins in* Opt*.*

(iv) *Suppose that for a* BF 3^+*-bin B, the first item in B is not huge, and no new bin is opened after opening B and before packing the third item into B. Then the first three items packed into B are a_{j-2}, a_{j-1} and a_j for some j (i.e., they are adjacent in I). Furthermore, these three items are packed into different bins in* Opt*.*

(v) *Suppose that a_j is the last item packed into a* BF *bin B. Then for all $j' > j$, we have $a_{j'} + s(B) > 1$ (i.e., no later items fit into B). Consequently, no later item has level $s(B)$ or larger in* BF *packing.* □

For the rest of the proof we assume that our instance I satisfies the properties from Lemma 4.1. The following lemma states the consequences for the common bins: The medium items are packed as in FirstFit and the small items are restricted to only first few common bins.

Lemma 4.2. (i) *Any item $a_j > 1/6$ packed into a regular bin C_i has the property that at the time of its packing, a_j does not fit into any previous common bin.*

(ii) *If a small item a_j is packed into a common bin, then this is a common bin with the largest level at the time of packing a_j. Except for C_1 and F, any small item in a common bin has level at least $2/3$.*

(iii) *From the moment when there are two common bins with level at least $2/3$ on, no small item arrives. In particular, no small item is packed into a common bin opened later than C_2.*

(iv) *If $a_j \in C_2$ is small, some $a_k > 1/6$, $k > j$ (i.e., a_k is after a_j), is packed into C_1.* □

In the next lemma we state some properties important for the freaky case. For the rest of the proof, let g_0 denote the item in bin G guaranteed by the next lemma. Note that the lemma implies that there are at least three items packed into G, as there are two other items in G when F opens.

Lemma 4.3. *In the freaky case, the* BF *packing satisfies the following:*
(i) *There exists an item g_0 that is packed into bin G such that g_0 arrives after f_2 and $s(F) + g_0 > 1$. Furthermore, $s(F) + s(G) > 1 + d_0$.*
(ii) *If the regular bins C_i and C_k are opened before F then $s(F) > 2/3$ and $s(C_i) + s(C_k) + s(F) > 2$.* □

Lemma 4.4. *In the* BF *packing the following holds:*
(i) *The total size of any $k \geq 2$* BF*-bins is greater than $k/2$.*
(ii) *If d_0 is defined, then $s(\mathcal{H} \cup \mathcal{D}) \geq (\delta + \eta)/2 + (\delta + \eta - 2)\Delta$.*
(iii) *The total number of huge-first and dedicated bins is $\delta + \eta \leq$* OPT.
(iv) *Suppose that C is a regular bin of size $s(C) = 2/3 - 2x$ for some $x \geq 0$. For any bin B before C we have $s(B) > 2/3 + x$ and for any regular or big bin B after C we have $s(B) > 2/3 + 4x$.*
(v) *Suppose we have a set $\mathcal{A}$ of k common and big bins such that there are at least 3 common bins among them. Then $s(\mathcal{A}) > 2k/3$.* □

Now we assume that the instance violates the absolute ratio 1.7 and derive some easy consequences that exclude some degenerate cases. Note that the values of $1.7 \cdot$ OPT are multiples of 0.1 and BF is an integer, thus BF $> 1.7 \cdot$ OPT implies BF $\geq 1.7 \cdot$ OPT $+ 0.1$. Typically we derive a contradiction with the lower bound $S \leq$ OPT on the optimum.

Lemma 4.5. *If* BF $> 1.7 \cdot$ OPT *then the following holds:*
(i) OPT ≥ 7.
(ii) *No common bin C has size $s(C) \leq 1/2$.*
(iii) *The total number of dedicated and huge-first bins is bounded by $\eta + \delta \geq 5$. If d_0 is not defined then there is no huge-first bin, i.e., $\eta = 0$.*
(iv) *The number of regular bins is at least $\rho \geq$* OPT$/2 + 1 > 4$. *If* BF $\geq 1.7 \cdot$ OPT $+ \tau/10$ *for some integer $\tau \geq 1$ then $\rho >$ (*OPT $+ \tau)/2$. □

The weight function, amortization, exceptional set

Now we give the modified and final definition of the weight function. The weight is modified only for d_0 and f_1 and their modified bonus is at least 0.1. Thus Lemma 2.2 still holds, as its proof uses at most 0.1 of bonus for each item.

Definition 4.6. *The weight function w, bonus $\overline{w}$ and scaled size $\overline{\overline{w}}$ are defined as follows:*

If d_0 is defined, we define $\overline{w}(d_0) = 0.4 - \frac{3}{5}\Delta$.
If f_1 is defined, we define $\overline{w}(f_1) = 0.1$

For any other item a, we define $$\overline{w}(a) = \begin{cases} 0 & \text{if } a \leq \frac{1}{6}\,, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } a \in \left[\frac{1}{6}, \frac{1}{3}\right]\,, \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right]\,, \\ 0.4 & \text{if } a > \frac{1}{2}\,. \end{cases}$$

For every item a we define $\overline{\overline{w}}(a) = \frac{6}{5}a$ *and* $w(a) = \overline{\overline{w}}(a) + \overline{w}(a)$.
For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for $\overline{\overline{w}}$ and $\overline{w}$. Furthermore, let $W = w(I)$ be the total weight of all items of the instance I.

Note that H-items are exactly the items with bonus greater than 0.1.

In the previous definition, the function $\overline{w}$ is continuous on the case boundaries, except for a jump at 0.4. Furthermore, if we have a set A of k items with size in $[\frac{1}{6}, \frac{1}{3}]$ and $d_0 \notin A$, then the definition implies that the bonus of A is exactly $\overline{w}(A) = \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$. More generally, if A contains at least k items and no H-item, then we get an upper bound $\overline{w}(A) \leq \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$.

The analysis of OPT-bins and big, dedicated and huge-first BF-bins in the next two lemmata is easy.

Lemma 4.7. *For every optimal bin A its weight $w(A)$ can be bounded as follows:*
(i) *$w(A) \leq 1.7$.*
(ii) *If A contains no H-item, then $w(A) \leq 1.5$.* □

Lemma 4.8. (i) *The total weight of the big bins is $w(\mathcal{B}) \geq \overline{\overline{w}}(\mathcal{B}) \geq \beta$.*
(ii) *The total weight of the dedicated and huge-first bins is $w(\mathcal{D} \cup \mathcal{H}) \geq \delta + \eta$.* □

The analysis of the common bins is significantly harder. Typically we prove that their weight is at least $\gamma - 0.2$ which easily implies that $\text{BF} \leq 1.7 \cdot \text{OPT} + 0.1$. Due to the integrality of BF and OPT, this implies our main result whenever $\text{OPT} \not\equiv 7 \pmod{10}$. To tighten the bound by the remaining 0.1 and to analyze the freaky case, we need to reserve the bonus of some of the items in the common bins instead of using it for amortization; this is possible if we still have two items in each regular bins whose bonus we can use. Now we define a notion of an exceptional set E, which contains these items with reserved bonus. In the freaky case, $g_0 \in E$, as its bonus is always needed to amortize for F. Other items are added to E only if $\text{OPT} \equiv 7 \pmod{10}$, depending on various cases.

Definition 4.9. *A set of items E is called an* **exceptional set** *if*
(i) *for each $i = 2, \ldots, \rho$, the bin C_i contains at least two items $c, c' > \frac{1}{6}$ that are not in E;*
(ii) *if $\text{OPT} \not\equiv 7 \pmod{10}$ then $E = \emptyset$ in the regular case and $E = \{g_0\}$ in the freaky case; and*
(iii) *if $\text{OPT} \equiv 7 \pmod{10}$ then E has at most two items and $g_0 \in E$ in the freaky case.*

The next lemma modifies the amortization lemma for the presence of the exceptional set.

Lemma 4.10. (i) *Let $i = 2, \ldots, \rho$ and $s(C_{i-1}) \geq 2/3$. Then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E) \geq 1$.*
(ii) *In the freaky case, if $s(F) \geq 2/3$ then $\overline{\overline{w}}(F) + \overline{w}(f_1) + \overline{w}(g_0) \geq 1$.*

Proof. **(i):** Let $c, c' > \frac{1}{6}$ be two items in $C_i \setminus E$; their existence is guaranteed by the definition of the exceptional set. By Lemma 4.2(i), $c, c' > 1 - s(C_{i-1})$. The claim follows by Lemma 2.2 (which applies even to the modified weights, as we noted before).

(ii): Lemma 4.3(i) implies $g_0 > 1 - s(F)$. Trivially, $f_1 > 1/2 > 1 - s(F)$. Thus we can apply Lemma 2.2 with $c = g_0$ and $c' = f_1$ and the claim follows. □

Analyzing the common bins

The following proposition is relatively straightforward if $s(C_\rho) \geq 2/3$, otherwise it needs a delicate argument. It implies easily our upper bound with the additive term 0.1.

Proposition 4.11. *Let* $\text{OPT} \geq 8$, $\text{BF} > 1.7 \cdot \text{OPT}$, *and* E *be an exceptional set. Then:*

(i) $w(\mathcal{R}) - \overline{w}(E) \geq \rho - 0.2$.
(ii) *If* C_ρ *has rank at least* 3 *then* $w(\mathcal{R}) - \overline{w}(E) \geq \rho$.
(iii) *In the freaky case, if* $E = \{g_0\}$, *and* $G = C_g \neq C_\rho$ *then we have* $w(\mathcal{R}) - \overline{w}(E) - \overline{\overline{w}}(C_g) - \overline{w}(C_{g+1}) \geq \rho - 1.2$. □

Proposition 4.12. *For any instance of bin packing with* $\text{OPT} \geq 8$, *we have* $W > \text{BF} - 0.2$ *and* $W \leq 1.7 \cdot \text{OPT}$. *Thus also* $\text{BF} \leq 1.7 \cdot \text{OPT} + 0.1$.

Proof. Suppose that $\text{BF} > 1.7 \cdot \text{OPT}$. First we show that $w(\mathcal{C}) \geq \gamma - 0.2$, distinguishing three cases.

In the regular case we set $E = \emptyset$ and Proposition 4.11(i) gives $w(\mathcal{C}) \geq \gamma - 0.2$.

In the freaky case, if $s(F) \geq 2/3$, we set $E = \{g_0\}$, then sum Lemma 4.10(ii) and Proposition 4.11(i) to obtain $w(\mathcal{C}) = w(\mathcal{R}) - \overline{w}(E) + \overline{w}(g_0) + w(F) > \rho - 0.2 + 1 = \gamma - 0.2$.

In the freaky case, if $s(F) < 2/3$, then Lemma 4.3(ii) implies that F opens before C_2 and $G = C_1$. Each C_j, $j \geq 2$, contains two items larger than 1/3, thus $w(C_j) > 1$. Finally, $f_1 < 2/3$, thus the level of C_1 when F opens is greater than 1/3. Using Lemma 4.3(i) we have $s(F) + g_0 > 1$, thus also $g_0 > 1/3$ and $\overline{w}(g_0) \geq 0.1$. Thus $w(G) + w(F) \geq \overline{\overline{w}}(G) + \overline{\overline{w}}(F) + \overline{w}(g_0) + \overline{w}(f_1) \geq \frac{6}{5}(\frac{1}{3} + 1) + 0.1 + 0.1 = 1.8$. Summing this with $w(C_j) > 1$ for $j \geq 2$ we obtain $w(\mathcal{C}) > \gamma - 0.2$ as well.

Together with Lemma 4.8, $w(\mathcal{C}) > \gamma - 0.2$ implies $W = w(\mathcal{B}) + w(\mathcal{D}) + w(\mathcal{H}) + w(\mathcal{C}) > \beta + \eta + \delta + (\gamma - 0.2) = \text{BF} - 0.2$. By Lemma 4.7(i) we have $W \leq 1.7 \cdot \text{OPT}$. Thus $\text{BF} - 0.2 < W \leq 1.7 \cdot \text{OPT}$. Since BF and OPT are integers the theorem follows. □

Now after having proved $\text{BF} \leq 1.7 \cdot \text{OPT} + 0.1$, we are going to prove our main result.

Theorem 4.13. *For any instance of bin packing we have* $\text{BF} \leq 1.7 \cdot \text{OPT}$.

Proof. Suppose the theorem does not hold. Then Proposition 4.12 implies $\text{BF} = 1.7 \cdot \text{OPT} + 0.1$ and integrality of OPT and BF then gives $\text{OPT} \equiv 7 \pmod{10}$, in particular OPT is odd.

In general, we try to save 0.1 in the analysis of the common bins, i.e., to prove $w(\mathcal{C}) > \gamma - 0.1$. In some of the subcases we need to use some additional weight of other bins and we then show $W > \mathrm{BF} - 0.1$. In both cases we then get $\mathrm{BF} - 0.1 < W \leq 1.7 \cdot \textsc{Opt}$ and the theorem follows by integrality of BF and Opt. In a few remaining cases we derive a contradiction directly.

The proof splits into three significantly different cases, $\textsc{Opt} = 7$, the regular case, and the freaky case. We give only a sketch of the proof in the freaky case. The next lemma enables the parity argument we mentioned before; it is thus also needed in the regular case.

Lemma 4.14. *Suppose that every* Opt*-bin contains an H-item. Then no* Opt*-bin contains two* 2*-items* c_1 *and* c_2*.* □

After excluding some easy subcases of the freaky case, we in particular know that every Opt-bin contains an H-item. The general idea of the proof in the freaky case is that we try to find an item c different from f_1 such that the bonus of $\{g_0, c\}$ is sufficient and can be used to pay for the freaky bin F. If we find such c, we save the bonus 0.1 of f_1 and use it to tighten Proposition 4.11 by the necessary 0.1. We have three subcases.

Case 1: F opens after C_2. Thus F contains no small item by Lemma 4.2(iii); since f_1 is huge and $s(F) < 5/6$, it follows that F is a 2-bin containing only f_1 and f_2.

The intuition is that we use the bonus of f_2 instead of f_1 to pay for F. However, in general, the bonus of $\{g_0, f_2\}$ is not sufficient to pay for F, if F is smaller than C_g. In that case, the bonus of $\{g_0, f_2\}$ is sufficient to pay for G_g and we use the bonus of the next common bin, G_{g+1} to pay for F. A further complication is that the bonus of $\{g_0, f_2\}$ smaller than necessary by a term proportional to Δ; this is compensated by the dedicated and huge-first bins.

Case 2: F opens before C_2 and some C_k, $k \geq 2$, has rank at least three. Let c be one of the medium items in this C_k and set $E = \{g_0, c\}$. Then E is a valid exceptional set. Furthermore, c does not fit into F.

If $s(F) \geq 2/3$, we have $\overline{\overline{w}}(F) + \overline{w}(E) \geq 1$ by Lemma 2.2. Using Proposition 4.11(i) we have $w(\mathcal{C}) \geq (w(\mathcal{R}) - \overline{w}(E)) + (\overline{\overline{w}}(F) + \overline{w}(E)) + \overline{w}(f_1) \geq \rho - 0.2 + 1 + 0.1 = \gamma - 0.1$.

If $s(F) < 2/3$, we have $\overline{w}(E) = 0.2$ and by Lemma 4.4(iv), C_ρ is a 2-bin such that $s(C_\rho) + s(F) > 4/3$. Thus $\overline{w}(E) + w(F) + \overline{\overline{w}}(C_\rho) > 0.2 + 0.1 + 1.6 = 1.9$. Adding all the inequalities $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E) \geq 1$, $i = 2, \ldots, \rho$ from Lemma 4.10(i), we get $w(\mathcal{C}) > \gamma - 0.1$.

Case 3: F opens before C_2 and each C_i, $i \geq 2$, has rank 2. Then all bins C_i, $i \geq 2$, are 2-bins and by Lemma 4.14, all items in these $\rho - 1$ bins are packed into different optimal bins. Thus there are at most $\textsc{Opt}/2$ such bins, and since Opt is odd (from $\textsc{Opt} \equiv 7 \pmod{10}$) we actually get $\rho \leq (\textsc{Opt} + 1)/2$. and thus $\gamma = \rho + 1 \leq \textsc{Opt}/2 + 3/2$. Instead of using the weights, here we get a contradiction by bounding the size of all the bins. □

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