The complexity of the lower envelope of segments with h endpoints

JIŘÍ MATOUŠEK PAVEL VALTR

Department of Applied Mathematics, Charles University
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
e-mail: {matousek,valtr}@kam.mff.cuni.cz

Abstract

For $h = 2^k$, we construct a set $\mathcal{L}$ of $\frac{h}{2}\log_2 h$ line segments in general position such that the segments of $\mathcal{L}$ have h endpoints in total and some part of each segment of $\mathcal{L}$ appears on the lower envelope of $\mathcal{L}$. It follows that a face in an arrangement of segments with h endpoints can have complexity $\Omega(h \log h)$, which is tight up to a constant factor.

1 Introduction

Arkin *et.al.* [AHKMN] have shown that the complexity of a single face in an arrangement of n line segments determined by h endpoints is $O(h \log h)$. For $n = \Omega(h \log h/\alpha(h))$, this improves the bound $O(n\alpha(n))$ following from the theory of Davenport-Schinzel sequences (e.g., see [SA]). Also a lower bound $\Omega(h\alpha(h))$ follows from the theory of Davenport-Schinzel sequences, as noticed in [AHKMN]. In this note we give an $\Omega(h \log h)$-lower bound, which is thus tight up to a constant factor. In fact, we show that there is an arrangement of line segments with h endpoints whose lower envelope has complexity $\Omega(h \log h)$.

Theorem 1 *For $h = 2^k$, there is a set $\mathcal{L}$ of $n = \frac{h}{2}\log_2 h$ line segments with h endpoints in general position, such that the lower envelope of $\mathcal{L}$ contains a part of each segment in $\mathcal{L}$.*

The bound in Theorem 1 is tight up to a constant factor, even if we allow x-monotone pseudosegments.

Theorem 2 *The lower envelope of a set of x-monotone pseudosegments with h endpoints in total is formed by parts of at most $h \log h$ of the pseudosegments.*

Theorem 1 is proved in Section 2, Theorem 2 in Section 2.

We say that a segment s *appears* on the lower envelope of a set $\mathcal{L}$ of segments, if a subsegment of s of positive length lies on the lower envelope of $\mathcal{L}$. For a set $\mathcal{L}$ of line segments (pseudosegments), we denote the set of all endpoints of segments (pseudosegments) of $\mathcal{L}$ by $E(\mathcal{L})$.

2 The construction

In this section we (inductively) construct a set $\mathcal{L}_k$ of $n = \frac{h}{2}\log_2 h = k2^{k-1}$ line segments with $h = 2^k$ endpoints in general position, such that the lower envelope of $\mathcal{L}_k$ contains a part of each segment in $\mathcal{L}_k$.

For $k = 1$, $\mathcal{L}_1$ contains one, non-vertical, line segment. Suppose now that $k > 1$ and that we have already constructed the set $\mathcal{L}_{k-1}$ with the above properties and with 2^{k-1} endpoints, whose x-coordinates are pairwise different.

Let c be a point which, say, shares a vertical line with some endpoint in $E(\mathcal{L}_{k-1})$ and has a much larger y-coordinate. Let $\mathcal{L}'_{k-1}$ be a set of segments centrally symmetric to $\mathcal{L}_{k-1}$ with the center of symmetry in c. Then each segment of $\mathcal{L}'_{k-1}$ appears on the upper envelope of $\mathcal{L}'_{k-1}$. Put $\mathcal{L} = \mathcal{L}_{k-1} \cup \mathcal{L}'_{k-1} \cup \mathcal{C}_{k-1}$, where $\mathcal{C}_{k-1}$ is the set of 2^{k-1} segments through c joining the points in $E(\mathcal{L}_{k-1})$ with their counterparts in $E(\mathcal{L}'_{k-1})$. Let h be the horizontal line through c. If c was taken with a sufficiently large y-coordinate, then there is a vertical line v left of each segment of $\mathcal{L}$, such that any line determined by a pair of points of $\mathcal{L}_{k-1}$ crosses v below h. Symmetrically, any line determined by a pair of points of $\mathcal{L}'_{k-1}$ crosses v above h. Let t be the point of intersection of the lines h and v.

Consider a projective transformation P which sends the line v to infinity and projects the other lines through t onto the vertical lines. Then P projects the lower envelope of $\mathcal{L}_{k-1}$ onto the lower or onto the upper envelope of $P(\mathcal{L}_{k-1})$. It is not difficult to find P so that it projects the lower envelope of $\mathcal{L}_{k-1}$ onto the lower envelope of $P(\mathcal{L}_{k-1})$. Then P projects the upper envelope of $\mathcal{L}'_{k-1}$ onto the lower envelope of $P(\mathcal{L}'_{k-1})$. Thus, all the $|\mathcal{L}_{k-1}| + |\mathcal{L}'_{k-1}| = (k-1)2^{k-1}$ segments of $P(\mathcal{L}_{k-1} \cup \mathcal{L}'_{k-1})$ appear on the lower envelope of $P(\mathcal{L})$. The 2^{k-1} segments of $P(\mathcal{C}_{k-1})$ meet in the point $P(c)$ which lies on the lower envelope of $P(\mathcal{L})$. Let $\mathcal{L}_k$ be the set $P(\mathcal{L})$, after the endpoints in $E(P(\mathcal{L}))$ were properly perturbed so that all the segments of $P(\mathcal{C}_{k-1})$ appear on the lower envelope of $\mathcal{L}_k \approx P(\mathcal{L})$. Then the lower envelope of $\mathcal{L}_k$ is formed by parts of all the $(k-1)2^{k-1} + 2^{k-1} = k2^{k-1}$ segments of $\mathcal{L}_k$. This shows Theorem 1.

3 The upper bound

???

References

[AHKMN] E.M. Arkin, D. Halperin, K. Kedem, J. Mitchell, and N. Naor. Arrangements of segments that share endpoints: single face results. ???

[SA] M. Sharir and P.K. Agarwal. *Davenport–Schinzel sequences and their geometric applications.* Cambridge Univ. Press 1995.