Integer points in rotating convex bodies

Imre Bárány
Jiří Matoušek

Abstract

Let K be a planar convex body symmetric about the origin. We define $P(K)$ as the probability of $\tau K \cap \mathbf{Z}^2 \neq \{0\}$, where $\tau \in SO(2)$ is a random rotation around the origin and $\mathbf{Z}^2$ denotes the integer lattice, and we let $P(v) = \inf\{P(K) : \mathrm{vol}(K) = v\}$. By Minkowski's theorem, $P(v) = 1$ for $v > 4$, and $P(v) = 0$ for $v < \pi$. We describe the behavior of $P(v)$ in the intervals $[\pi, \pi + \varepsilon_0]$ and $[4 - \varepsilon_0, 4]$ for a small positive constant ε_0.

1 Introduction and results

Let K be a bounded open convex body in the plane symmetric about 0. Let $\mathrm{vol}(K)$ denote its area. Let τ be a rotation around 0 by a random angle (uniformly distributed in $[0, 2\pi)$). We put

$$P(K) = \mathrm{Prob}\left[\mathbf{Z}^2 \cap \tau K \neq \{0\}\right],$$

and for $v \geq 0$, we define

$$P(v) = \inf\{P(K) : \mathrm{vol}(K) = v\}.$$

We have $P(\pi) = 0$, as is witnessed by the open unit disc, and Minkowski's classical theorem ([Min96]) implies that $P(v) = 1$ for any $v > 4$. Our goal is to understand the behavior of the function $P(v)$ on the interval $[\pi, 4]$, in particular, near the endpoints of this interval. We will show, for instance, that for $v = 4 - \varepsilon$, $P(v)$ behaves like $1 - const \cdot \sqrt{\varepsilon}$. This is the interval where the condition of Minkowski's theorem just fails to hold by a small ε. So our result says that in this case the conclusion of Minkowski's theorem fails to hold with probability at most $O(\sqrt{\varepsilon})$.

Areas close to π. The following theorem describes the behavior of $P(v)$ in the range $\pi \leq v \leq \pi + \varepsilon_0$ for a suitable positive constant $\varepsilon_0 > 0$. For a given $\varepsilon \in (0, \varepsilon_0)$, let K_ε denote the open convex body indicated in Fig. 1. This K_ε is very close to the unit disc. It is symmetric under rotation by $\frac{\pi}{2}$, and the segments AB, BC, CD, and DE all have equal length, which is set

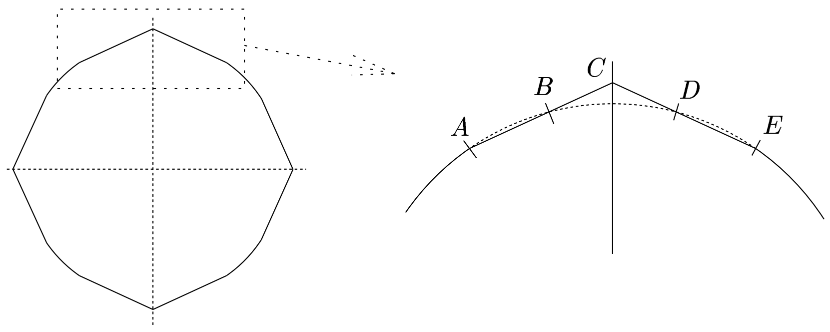


Figure 1: The body K_ε.

in such a way that $\mathrm{vol}(K_\varepsilon) = \pi + \varepsilon$. An elementary computation shows that $\alpha \sim \sqrt[3]{3\varepsilon}$ as $\varepsilon \to 0$, where α is the length of the arc BD and $f \sim g$ means $\lim \frac{f}{g} = 1$. It follows that

$$P(K_\varepsilon) \sim \tfrac{2}{\pi}\sqrt[3]{3\varepsilon} = (0.918\ldots)\varepsilon^{1/3}.$$

Theorem 1.1 *There is a positive constant ε_0 such that for $\varepsilon \in [0, \varepsilon_0]$,*

$$P(\pi + \varepsilon) = P(K_\varepsilon).$$

Moreover, if $\mathrm{vol}(K) = \pi + \varepsilon$ *and $P(K) = P(K_\varepsilon)$ then K is a rotated copy of K_ε, i.e. K_ε is the unique extremal body.*

Areas close to 4. For areas close to 4, we do not characterize the extremal bodies, but most likely they are simply squares. Let Q denote the square $[-1, 1]^2$; then the square $Q' = \sqrt{1 - \frac{\varepsilon}{4}}Q$ has area $4 - \varepsilon$ and

$$P(Q') = 1 - \tfrac{4}{\pi} \arccos \sqrt{1 - \tfrac{\varepsilon}{4}} \approx 1 - \tfrac{2}{\pi}\sqrt{\varepsilon}.$$

We believe that Q' is the extremal body for small enough ε. The order of magnitude of $1 - P(4 - \varepsilon)$ is given by the next result.

Theorem 1.2 *There exist constants $\varepsilon_0 > 0$ and $c_1, c_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have*

$$1 - c_1\sqrt{\varepsilon} \le P(4 - \varepsilon) \le 1 - c_2\sqrt{\varepsilon}.$$

The proofs of both theorems use elementary geometric arguments. The lower bound in Theorem 1.2 is shown by Q'. For the upper bound we have to use something like Minkowski's theorem, and indeed in one of the stages of the proof we argue similarly to the usual proof of Minkowski's theorem, see e.g. [PA95] or [Gru95].

Both proofs have analogous global structure: we treat separately thin bodies K, showing that $P(K)$ is much larger for them than the optimal value. In these arguments, we estimate the contribution to $P(K)$ of an individual lattice point and then we sum up over all lattice points.

In the second part of each proof, we consider bodies with not too large circumradius. For Theorem 1.1, having established the existence of a body attaining the infimum, we proceed by local modifications of the boundary that increase the volume and do not increase $P(K)$. This is somewhat similar in spirit to proofs of isoperimetric inequalities by symmetrizations, for example. Since the extremal body is not completely simple, several types of modifications are needed.

For Theorem 1.2 the modification strategy does not quite work for bodies that are close to the square because there are too many cases to consider with too much computation. This is why we couldn't characterize the extremal bodies. So we have a different strategy for bodies that are not thin: we show that such a K can be closely approximated by a "tile". These tiles are parallelograms or hexagons and they form nice lattice packings, although they do not necessarily fill the whole plane. Then we analyze several possible shapes of the tiles and show that in every case $P(K) \geq 1 - O(\sqrt{\varepsilon})$.

Unfortunately, the proofs are quite lengthy and contain many tedious details. It would be nice to find shorter proofs. Also, determining the extremal body for areas close to 4 seems feasible but at the price of a detailed, lengthy, and exhausting case analysis.

Another interesting aspect would be investigating the stability of the inequalities in the above theorems. The question is similar to Bonnesen's inequality (see [Bon29] or the survey by Groemer [Gro95]). Assume K is 0-symmetric, has area $\pi + \varepsilon$ and $P(K) < const \cdot \varepsilon^{\frac{1}{3}}$. Does it follow that K is close to the unit disc? Or more quantitatively: how close is then K to the unit disc?

A further open problem is extending the results to higher dimensions. As the planar case indicates, characterization of extremal bodies may be quite difficult or even out of reach, and even determining the correct orders of magnitude of $P(K)$ near the critical values ($\mathrm{vol}(B^d)$ and 2^d in dimension d) seems challenging at present. It looks likely that for small enough ε

$$P(\mathrm{vol}(B^d) + \varepsilon) = O\left(\varepsilon^{\frac{d(d-1)}{2(d+1)}}\right) \text{ and } P(2^d - \varepsilon) = 1 - O\left(\varepsilon^{\frac{d(d-1)}{4}}\right);$$

these inequalities are suggested by examples similar to K_ε and Q'.

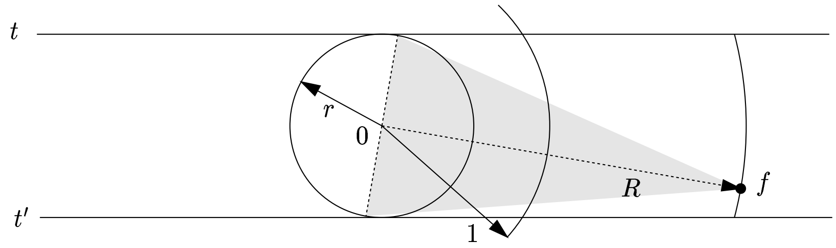


Figure 2: The inscribed and circumscribed circles.

2 Notation and preliminaries

In the sequel, by a *body*, we mean a symmetric open convex body in the plane.

Let S^1 denote the unit circle. We identify the set of all rotations around the origin with S^1. For a body K and $z \in \mathbf{Z}^2$, let $T_z(K) = \{\tau \in S^1 : z \in \tau K\}$ and $T(K) = \bigcup_{z\in\mathbf{Z}^2\setminus\{0\}} T_z(K)$; thus, $P(K) = \mu(T(K))$, where μ is the one-dimensional Lebesgue measure on S^1 scaled so that $\mu(S^1) = 1$.

The following notation concerning various parameters of the body K will be used. Let R denote the radius of the smallest enclosing circle of K, and let r be the radius of the largest circle inscribed in (the closure of) K (see Fig. 2). Let f be a point of the boundary ∂K at distance R from 0. The largest inscribed circle touches ∂K at some two diametrically opposite points, and it has tangents t and t' common with K at these points. In the figure, K is thus contained in the horizontal strip between t and t', and we have

$$2rR \leq \mathrm{vol}(K) \leq 4rR;$$

the second inequality is obvious from the containment and the first one follows by considering the gray triangle, which must be contained in K.

Minimal bodies. Call a body K *minimal* if $P(K) = P(\mathrm{vol}(K))$. For some of the subsequent arguments, we will need the existence of minimal bodies. We prove it by a compactness argument, but such an argument needs a bound on the circumradius of the considered bodies. Later on, we will show that only bodies with a limited circumradius can come close to being minimal. With such a bound on the circumradius, we will be able to use the following lemma.

Lemma 2.1 *For $v \in (\pi, 4)$ and $R \geq 1$, let $\mathcal{K}(v,R)$ denote the set of all symmetric open convex bodies of area v and circumradius at most R. Whenever $\mathcal{K}(v,R) \neq \emptyset$, there exists a $K_{min} \in \mathcal{K}(v,R)$ such that $P(K_{min}) = \inf_{K\in\mathcal{K}(v,R)} P(K)$.*

Proof. We note that if the circumradius is bounded then the inradius r is bounded away from 0.

Write $\mathcal{K} = \mathcal{K}(v, R)$, and equip $\mathcal{K}$ with the Hausdorff metric h: $h(K, K') = \min(h_1(K, K'), h_1(K', K))$, where $h_1(K, K') = \sup_{x \in K} \inf_{y \in K'} |x - y|$. This $\mathcal{K}$ is a compact metric space (use the Arzelà–Ascoli theorem; this is a similar argument as in the Blaschke selection theorem).

Let $P = \inf_{K \in \mathcal{K}} P(K)$. There is a sequence $\{K_n\}_{n=1}^{\infty}$ of bodies in $\mathcal{K}$ with $P(K_n) \to P$, and by compactness, we may assume that it converges in the Hausdorff metric. We show that the limit body, which we denote by K_{min}, has the required property.

Let $\tau \in T(K_{min})$. Let z be an integer point with $z \in \tau K_{min}$. Since K_{min} is open, a neighborhood of z is contained in τK_{min} and we also have $z \in \tau K_n$ for all sufficiently large n. Consequently, we get

$$T(K_{min}) \subseteq \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} T(K_i).$$

Write $A_n = \bigcap_{i=n}^{\infty} T(K_i)$. Since $\mu(T(K_n)) \to P$, we obtain $\mu(A_n) \leq P$. We have $A_1 \subseteq A_2 \subseteq \cdots$, and by the σ-additivity of μ, we obtain $P(K_{min}) = \mu(\bigcup_{n=1}^{\infty} A_n) \leq P$. $\square$

3 Proof of Theorem 1.1

We consider bodies K with $\mathrm{vol}(K) \in (\pi, \pi + \varepsilon_0]$ for a small constant $\varepsilon_0 > 0$. We divide the proof into two parts. In the first part, we show that bodies K with large circumradius satisfy $P(K) \geq c > 0$ for an absolute constant c. Therefore, we can disregard them, since $P(K_\varepsilon) \to 0$ as $\varepsilon \to 0$ for the extremal body K_ε.

3.1 The thin case

Here we consider bodies with $R \geq \sqrt{2}$. For this case, we prove the following more precise result:

Proposition 3.1 *If $R \geq \sqrt{2}$ and $0 < \mathrm{vol}(K) < 4$, then $P(K) \geq c \cdot \mathrm{vol}(K)$ for a suitable absolute constant $c > 0$. (In fact, the result holds for any fixed $R > 1$ with a suitable $c = c(R)$.)*

Proof. Let

$$Z = \{z \in \mathbf{Z}^2 \setminus \{0\} : |z| \leq \tfrac{1}{\sqrt{2}} R, \gcd(z_1, z_2) = 1\}$$

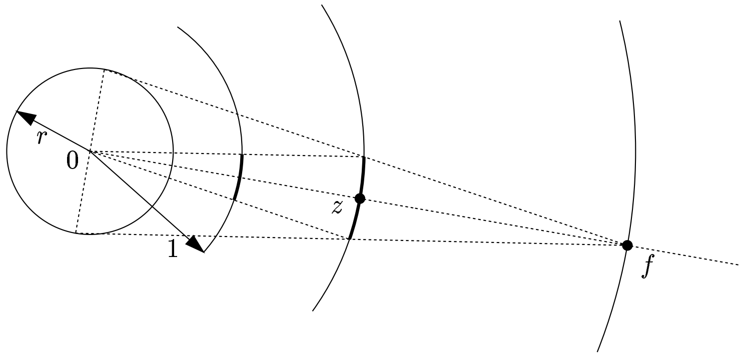


Figure 3: Rotations with $z \in \tau K$.

denote the set of all primitive integer vectors of length at most $\frac{1}{\sqrt{2}}R$. For each $z \in Z$, let u_z be the arc on the unit circle centered at $\frac{z}{|z|}$ and of length $\ell_z = \frac{r}{2|z|}$.

We claim that for all rotations τ that bring the point $\frac{f}{|f|}$ into the arc u_z, we have $z \in \tau K$; see Fig. 3. To this end, it suffices to show that u_z is contained in the thick arc on the unit circle in the figure. Simple calculation shows that the length of the thick arc is actually at least $2\frac{r(R-|z|)}{R\cdot|z|} \geq \frac{r}{2|z|}$ (as $|z| \leq \frac{1}{\sqrt{2}}R$).

Next, we claim that all the arcs u_z, $z \in Z$, are pairwise disjoint. Assume that $z, w \in Z$ are consecutive in the angular order around the origin; we need to show that the angle $\alpha = \angle z0w$ is at least $\frac{1}{2}(\ell_z + \ell_w)$. Since z and w are consecutive primitive vectors, the triangle $z0w$ contains no integer point and so it has area $\frac{1}{2}$ (Pick's theorem); at the same time, the area is $\frac{1}{2}|z| \cdot |w| \cdot \sin\alpha \leq \frac{1}{2}|z| \cdot |w| \cdot \alpha$. So $\alpha \geq \frac{1}{|z|\cdot|w|}$. The desired inequality $\alpha > \frac{1}{2}(\ell_z + \ell_w)$ reduces to $r(|z| + |w|) < 4$, which holds in view of $|z|, |w| < R$ and $2rR \leq \mathrm{vol}(K) < 4$. The disjointness of the arcs u_z is proved.

We now calculate

$$P(K) \geq \sum_{z \in Z} \frac{r}{2|z|} \geq c_1 r R \geq c\,\mathrm{vol}(K)$$

for a sufficiently small $c > 0$. This proves Proposition 3.1. □

3.2 The round case

Proposition 3.1 shows that for area $v \leq \pi + \varepsilon_0$, all bodies close to being minimal have circumradius $\leq \sqrt{2}$. By Lemma 2.1, we know that minimal bodies exist for these v. So we assume that K is a minimal body, necessarily

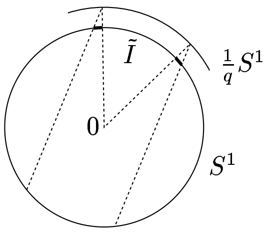


Figure 4: Shrinking makes $P(\tilde{K})$ smaller.

of circumradius at most $\sqrt{2}$, and we derive a series of claims about the shape of K. We already know that $P(K) \le P(K_\varepsilon) = O(\varepsilon^{1/3})$. Our basic tool for establishing further properties of a minimal K is the following observation.

Lemma 3.2 *Let $K, \tilde{K}$ be bodies of circumradius at most $\sqrt{2}$ and such that $P(\tilde{K}) \le P(K) < 1$ and $\mathrm{vol}(\tilde{K}) > \mathrm{vol}(K)$. Then K is not minimal.*

Proof. Let $L = q\tilde{K}$ where $q < 1$ is a scaling factor such that $\mathrm{vol}(L) = \mathrm{vol}(K)$; it suffices to check that $P(L) < P(\tilde{K})$.

Consider the set $T_{e_1}(\tilde{K})$ of all rotations τ bringing the point $e_1 = (0,1)$ into $\tau\tilde{K}$. This set is a union of open intervals, and it is easy to check that $T_{e_1}(L) \subseteq T_{e_1}(\tilde{K})$ and, moreover, for each interval $\tilde{I}$ of $T_{e_1}(\tilde{K})$, sufficiently small parts of $\tilde{I}$ near the endpoints (indicated by thick arcs in Fig. 4) are disjoint with $T_{e_1}(L)$. Since $T(\tilde{K})$ is not the whole S^1 by the assumption $P(\tilde{K}) \le P(K) < 1$, the inclusion $T(L) \subset T(\tilde{K})$ is proper, and we have $P(L) < P(\tilde{K})$. □

Note that the assumption $P(\tilde{K}) \le P(K)$ is satisfied if $\tilde{K} \cap S^1 \subseteq K \cap S^1$ or, more generally, if $\tilde{K} \cap S^1 \subseteq (K \cap S^1) \cup E$ for a set E of measure 0.

Next, we introduce some terminology. The boundary of K is partitioned into

- *circle arcs*, which are the connected components of $S^1 \cap \partial K$,
- *inside arcs*, which are the connected components of the part of ∂K lying strictly inside S^1, and
- *outside arcs*, which are the connected components of the part of ∂K lying strictly outside of S^1.

Note that the inside and outside arcs are open intervals in ∂K, and there are at most countably many of them.

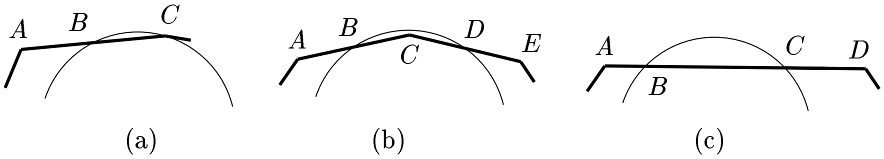


Figure 5: The cases in condition (P3).

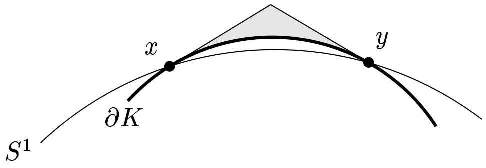


Figure 6: Replacing an outside arc.

Lemma 3.3 *A minimal body K, $\pi < \mathrm{vol}(K) \leq \pi + \varepsilon_0$, has the following properties:*

(P1) Each outside arc of K consists of two segments.

(P2) Each inside arc of K consists of one or two segments.

(P3) Each segment of an inside arc continues by a collinear segment of an outside arc in at least one direction; the possible cases are shown in Fig. 5. In case (a), we have $|AB| = |BC|$, in case (b) $|AB| = |BC|$ and $|CD| = |DE|$, and in case (c), $|AB| + |CD| \geq |BC|$.

Proof. If there is an outside arc, with endpoints x and y (as in Fig. 6), that is not made of two segments, we can produce a body $\tilde{K}$ by replacing the arc by two segments tangent to ∂K at x and at y as in the figure. The circumradius of $\tilde{K}$ is below $\sqrt{2}$, because the considered outside arc is short (of length $O(\varepsilon_0^{1/3})$) and K contains the inscribed circle of radius at least $\frac{1}{3}$. Of course, the arc symmetric to the considered arc xy is transformed in the same way, to make $\tilde{K}$ symmetric; the modifications in all the subsequent steps are made symmetrically as well (without mentioning this explicitly). By Lemma 3.2, we get that K is not minimal unless it has property (P1).

To get property (P2), we proceed similarly. An inside arc with endpoints x and y as in Fig. 7 is replaced by two segments tangent to ∂K at x and at y; we obtain either a single inside arc (as in the left figure) or two inside arcs and a new circle arc (as in the right figure). Note that this works even if, say, the tangent segment to ∂K at x is tangent to S^1; in such case, we let the boundary of the modified body continue with an arc of S^1.

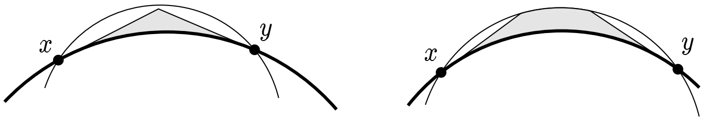


Figure 7: Transforming an inside arc.

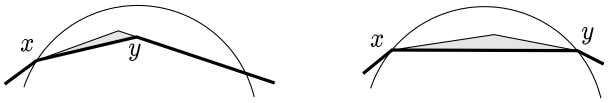


Figure 8: An inside arc that does not extend.

As for (P3), suppose that a segment $s = xy$ of an inside arc does not extend to either side within ∂K, as in one of the cases in Fig. 8. Then there are tangents to ∂K at x and at y making a positive angle with s (we already know that outside arcs consist of two segments!), and we can enlarge K to $\tilde{K}$ as indicated.

Thus, ∂K looks like one of the cases in Fig. 5 near each inside arc, and it remains to check the claimed equalities and inequalities concerning the lengths of the segments. For example, in case (a), if $|AB| < |BC|$, a suitable body $\tilde{K}$ can be produced by adding the gray triangle and removing the dashed triangle in Fig. 9(a). Appropriate modifications of K in cases (b) and (c) are indicated in figures (b) and (c). This proves Lemma 3.3. $\Box$

Lemma 3.4 *A minimal body K can be rotated in such a way that it is contained in the square $[-1,1]^2$ and the sides of the square are tangent to ∂K at their midpoints $\pm e_1$ and $\pm e_2$ (Fig. 10).*

Proof. By the equalities and inequalities in condition (P3), we see that the angular length of each inside arc is at most a constant multiple of the angular lengths of the adjacent outside arcs. The total angular length of the outside arcs is at most $P(K) = O(\varepsilon_0^{1/3})$. If $C = \partial K \cap S^1$ denotes the union of the circle arcs, we thus have $\mu(C) > \frac{3}{4}$, and so there is a rotation of K such that C is dense in some neighborhood of each of the four integer points on S^1. For such a rotation, the tangents to S^1 at $\pm e_1$, $\pm e_2$ are tangents to ∂K as well. $\Box$

From now on, we assume that the minimal body K is rotated as in Lemma 3.4. Let s_{++}, s_{+-}, s_{--}, s_{-+} denote the arcs of S^1 in the first, second, third, fourth quadrant, respectively, and similarly, let $k_{++}, \ldots, k_{-+}$ be the arcs of ∂K. Let K_1 be the body bounded by k_{++}, s_{+-}, k_{--}, and

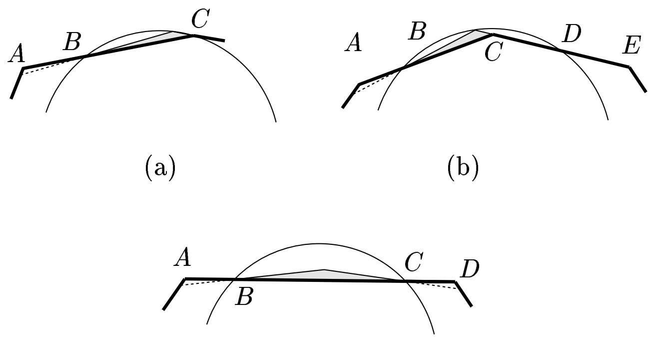


Figure 9: Proving the inequalities in (P3).

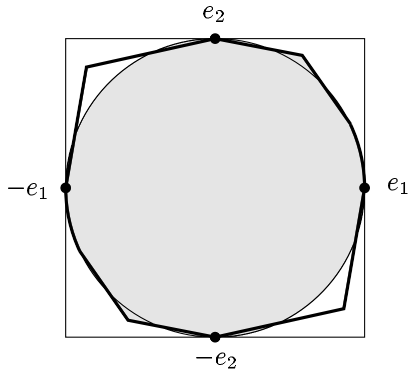


Figure 10: Placing K into the square.

s_{-+} (indicated by gray shading in Fig. 10), and let K_2 be bounded by the complementary arcs s_{++}, k_{+-}, s_{--}, and k_{-+}.

Let $\tau_{\pi/2}$ denote the counterclockwise rotation by $\frac{\pi}{2}$. Let $\varepsilon_i = \mathrm{vol}(K_i) - \pi$, $i = 1, 2$. We have $\varepsilon = \mathrm{vol}(K) - \pi = \varepsilon_1 + \varepsilon_2$ and $P(K) \geq \max(P(K_1), P(K_2))$, with equality only if $K_1 \cap S^1 \subseteq \tau_{\pi/2}(K_2 \cap S^1)$ or $K_2 \cap S^1 \subseteq \tau_{\pi/2}(K_1 \cap S^1)$. If $\varepsilon_1 > \varepsilon_2$, let $\tilde{K} = K_1 \cup \tau_{\pi/2} K_1$. Then $\mathrm{vol}(\tilde{K}) = 2\varepsilon_1 + \pi > \mathrm{vol}(K)$ and $P(\tilde{K}) = P(K_1) \leq P(K)$, and so K cannot be minimal. Therefore $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ and K_1 minimizes $\mu(K_1 \cap S^1)$ among the bodies with area $\pi + \frac{\varepsilon}{2}$ containing the arcs s_{+-} and s_{-+} in their boundary. Moreover, after possibly interchanging K_1 and K_2, we may assume $K \cap S^1 \subseteq (K_1 \cap S^1) \cup \tau_{\pi/2}(K_1 \cap S^1)$.

Lemma 3.5 *A minimal body K with* $\mathrm{vol}(K) = \pi + \varepsilon \leq \pi + \varepsilon_0$ *has the following properties:*

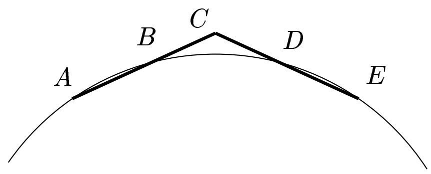


Figure 11: The only possible type of outside arcs.

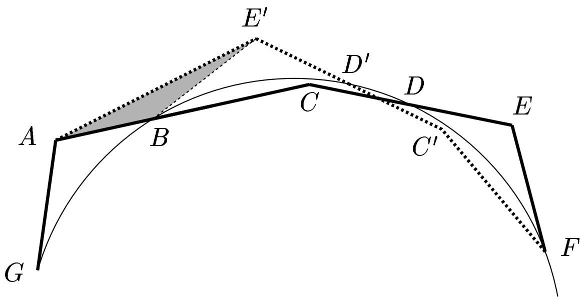


Figure 12: Excluding case (b).

(P4) No inside arc of K is as in cases (b) or (c) in Fig. 5.

(P5) Each of the two segments of an outside arc continues by a collinear segment of an inside arc.

(P6) Each of the two segments of each outside arc continues by a single-segment inside arc as in Fig. 11, where $|AB| = |BC| = |CD| = |DE|$.

Proof. Concerning (P4), we consider the situation as in case (b), and we suppose that $|0E| \geq |0A|$ (if not, we proceed symmetrically). We choose the point $D' \in S^1$ as in Fig. 12 so that the arcs BD' and DF on S^1 have the same lengths, and we let the polygonal line $E'C'F$ be a mirror reflection of the polygonal line ECB. First we construct a new body $\tilde{K}_1$ from K_1 by replacing the portion $ABCDEF$ of ∂K_1 by $AE'D'C'F$. The angle GAE' is convex since $|E'0| \geq |A0|$. We have $\mu(\tilde{K}_1 \cap S^1) = \mu(K_1 \cap S^1)$ and $\mathrm{vol}(\tilde{K}_1)$ exceeds $\mathrm{vol}(K_1)$ by the area of the triangle ABE'. Then we put $\tilde{K} = \tilde{K}_1 \cup \tau_{\pi/2}\tilde{K}_1$; we have $\mathrm{vol}(\tilde{K}) > \mathrm{vol}(K)$ and $P(\tilde{K}) \leq P(K)$. Hence K was not minimal by Lemma 3.2. Case (c) can be excluded by an analogous construction, and so (P4) holds.

Next, we consider a segment of an outside arc with endpoint $x \in S^1$ and we check (P5). We distinguish two cases. First, suppose that there are points of $S^1 \setminus K$ on the left of x and arbitrarily close to it, as in Fig. 13 left. Then

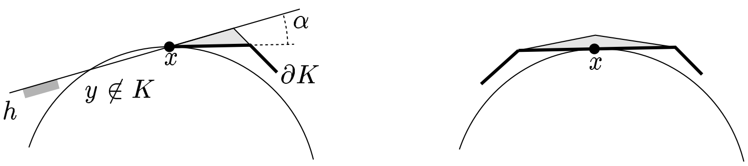


Figure 13: The proof of property (P5).

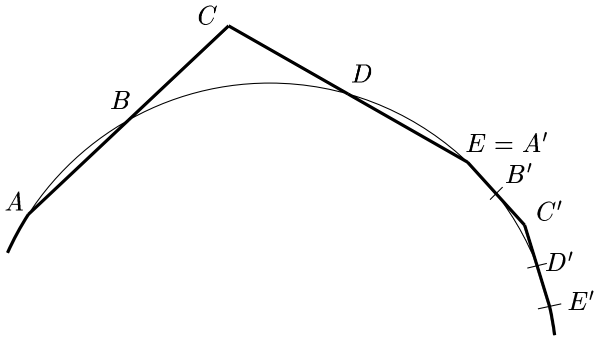


Figure 14: Two adjacent dents.

we form a new body $\tilde{K}_1$ as in the left figure, by intersecting K_1 with the (open) halfplane h and adding the gray triangle. If the angle α is sufficiently small, we have $\mathrm{vol}(\tilde{K}_1) > \mathrm{vol}(K_1)$ and $\tilde{K}_1 \cap S^1 \subseteq K_1 \cap S^1$. The argument is finished as the one above for the condition (P4).

Next, the situation can be as in the right figure, with segments of outside arcs of ∂K_1 emanating from x in both directions. Then $\tilde{K}_1$ can be produced by adding the indicated triangle. In this way, $\tilde{K}_1 \cap S^1$ has the extra point x, but Lemma 3.2 still applies.

We have already shown that each outside arc BCD continues with inside arcs on both sides, each formed by a single segment, so the situation is as in Fig. 11, with $|AB| = |BC|$ and $|CD| = |DE|$; this already implies $|BC| = |CD|$ as well. $\square$

In order to prove Theorem 1.1, it now suffices to verify that K has only one outside arc in the first quadrant. Call the portion of ∂K consisting of an outside arc and the two adjacent inside arcs, i.e. from A to E in Fig. 11, a *dent*. Suppose that there are at least two dents. The dents can be moved and interchanged along ∂K_1 without changing the area or $P(K_1)$. Thus, we may suppose that the considered two dents are adjacent as in Fig. 14. But then the construction as in Fig. 12 used above for proving (P4) can be applied to the portion from B to D' and we get non-minimality of K. Theorem 1.1 is proved. □

4 Proof of Theorem 1.2

The proof consists of three parts. First (Section 4.1) we treat bodies with relatively large circumradius $R \geq 3.1$ (thin bodies). Next (Section 4.2), we deal with bodies that have circumradius below 3.1 but are not very close to the square. For both these cases, we show that, for $\mathrm{vol}(K) = 4 - \varepsilon$, $P(K) \geq 1 - O(\varepsilon)$ and so these bodies cannot be minimal. In the last part (Section 4.3), we consider bodies with small Hausdorff distance to the square and use tiles and their floating bodies to finish the proof.

4.1 Thin bodies

This part is devoted to the proof of the following:

Proposition 4.1 *Let K be a body with $\mathrm{vol}(K) = 4 - \varepsilon$ and circumradius $R \geq 3.1$ and let $\varepsilon < \varepsilon_1$ for a sufficiently small $\varepsilon_1 > 0$. Then $P(K) \geq 1 - c\varepsilon$ for a positive constant $c > 0$. Thus, such K cannot be minimal.*

For the time being we only assume $R > 2.2$, since some of the auxiliary results obtained in the proof will be useful later. Note that this implies $r < 2/R < 0.91$.

Let $\mathrm{vol}(K) = 4 - \varepsilon$ and suppose that K has been rotated so that $K \cap \mathbf{Z}^2 = \{0\}$. Let h be a vector of length $\frac{\varepsilon}{R}$ orthogonal to $0f$, where, as before, f is a point on ∂K where the circumradius is attained. Set

$$K^+ = K + [-h, h].$$

Then $\mathrm{vol}(K^+) \geq 4 - \varepsilon + 4\varepsilon = 4 + 3\varepsilon > 4$, and so there is at least one nonzero integer point in $K^+ \setminus K$. Call a point $z \in K^+ \setminus K$ *moderate* if there is a line ℓ passing through z and avoiding K that intersects the line $0f$ at angle at most $\frac{\pi}{4}$ (Fig. 15).

Lemma 4.2 *If $R \geq 2.2$ then there is at least one moderate integer point in $K^+ \setminus K$.*

Proof. As was noted above, $K^+ \setminus K$ contains some nonzero integer point z, as well as $-z$. Choose the notation so that z is no farther from f than $-z$. Assume that z is not moderate. Choose a line ℓ separating z from K, let ℓ^+ be the closed halfplane determined by ℓ and not containing 0, and let $K^{\pm} = K^+ \setminus (\ell^+ \cup (-\ell^+))$ (so we cut off by ℓ and by its symmetric copy). We

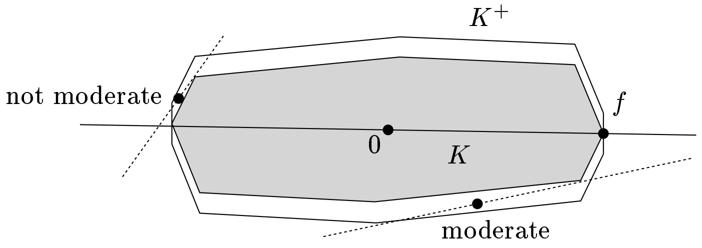


Figure 15: The body K^+ and the definition of a moderate point.

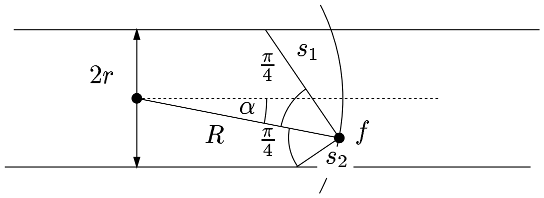


Figure 16: Estimating s_1 and s_2.

show first that $\mathrm{vol}(K^{\pm}) > 4$. As $\mathrm{vol}(K^+) > 4 + 3\varepsilon$ it suffices to show that $\mathrm{vol}(K^+ \cap \ell^+) < \varepsilon$.

Let α be the angle between the ray $0f$ and the parallel strip of width $2r$ enclosing K. Then $\sin\alpha \leq \frac{r}{R}$ (see Fig. 16). We may assume, by symmetry, that f is below the middle line of the strip. We estimate first the lengths, s_1 and s_2, of the segments indicated in Fig. 16.

Using $R > 2.2$ and $r < \frac{2}{R}$, it is easy to see that s_2 is the longest when $\alpha = 0$, and then it is equal to $\sqrt{2}r$. So $s_2 \leq \sqrt{2}r$. For s_1, we calculate

$$s_1 = \frac{r + R\sin\alpha}{\sin(\alpha + \frac{\pi}{4})} \leq R \cdot \frac{\frac{1}{2} + \sin\alpha}{\sin(\alpha + \frac{\pi}{4})}.$$

The right-hand side is maximized for the maximum possible α, i.e. the one with $\sin\alpha = \frac{r}{R} \leq \frac{1}{2}$, in which case it yields $s_1 < 1.04R$.

The intersection of ℓ with K^+ is a segment whose length $s \leq \max(s_1 + |h|, s_2 + |h|) \leq 1.1R$ (see Fig. 17). The cut-off area of $K^+ \cap \ell^+$ is at most s times the width of the strip between ℓ and $\ell + h$, which is at most $|h|\cos\frac{\pi}{4}$. Thus

$$\mathrm{vol}(K^+ \cap \ell^+) \leq s|h|\cos\frac{\pi}{4} \leq 1.1R \cdot \frac{\varepsilon}{R} \cdot \frac{1}{\sqrt{2}} < \varepsilon.$$

We have shown that $\mathrm{vol}(K^{\pm}) > 4$. Consequently $K^{\pm}$ contains another nonzero integer point w. We again choose the sign in such a way that w is

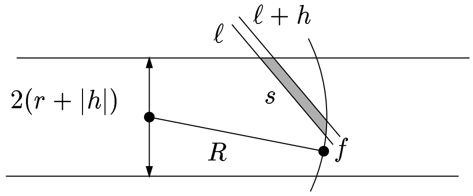


Figure 17: Bounding the portion cut off by ℓ.

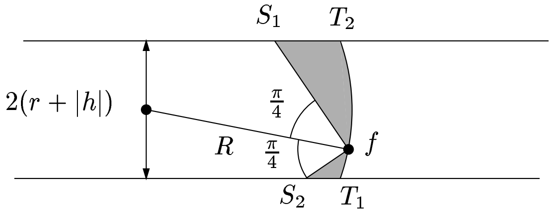


Figure 18: Bounding the distance of z and w.

closer to f than $-w$.

If w is moderate we are done, so we assume that w is not moderate. It follows that both z and w lie in the region indicated in Fig. 18 whose diameter is either S_1T_1 or S_2T_2. The segment S_2T_2 is the longest when $\alpha = 0$, and then $|S_2T_2| < \sqrt{5}(r + |h|) < \sqrt{5}$. As for S_1T_1, we have $|S_1T_1| \leq (1 + O(h)) \cdot \sqrt{4r^2 + |S_1T_2|^2}$ and $|S_1T_2| \leq (r + R\sin\alpha)\cot(\alpha + \frac{\pi}{4})$. Using $\sin\alpha \leq \frac{r}{R}$, we obtain $|S_1T_2| \leq 2r$ and it follows that $|S_1T_1| < \sqrt{8}$ (in fact, more detailed calculations reveal that it is even at most $\sqrt{5}$, which eliminates Case 3 below). Consequently $|z - w| < \sqrt{8}$.

We have to consider the following cases: $z - w$ equals $\pm e_1$ or $\pm e_2$ (Case 1), or $\pm e_1 \pm e_2$ (Case 2), or $\pm 2e_1 \pm e_2$ or $\pm e_1 \pm 2e_2$ (Case 3), or $\pm 2e_1$ or $\pm 2e_2$ (Case 4).

Since the convex quadrilateral $zw(-z)(-w)$ is contained in K^+ and it consists of four triangles of the same area as the lattice triangle $0zw$, it follows that the area of $0zw$ is at most 1. Therefore the area of $0zw$ equals either 1 or $\frac{1}{2}$. The area of $0zw$ is 1 if it contains no lattice point in its interior and has exactly one lattice point on its boundary (apart from its vertices). So the area of $0zw$ is 1 if and only if Case 4 occurs.

Case 1 We can assume, by symmetry, that $z = w + e_1$ with w having nonnegative x-component. As the area of $0zw$ is $\frac{1}{2}$, the y-component of z and w is ± 1, see Fig. 19. Since $\angle 0fw \leq \frac{\pi}{4}$, f lies in the union of the two discs indicated in the figure. At the same time, f is outside the disc of

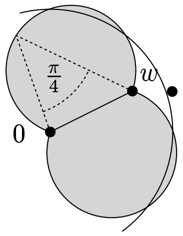


Figure 19: The case $z = w + e_1$.

radius $|z| - |h|$ centered at 0. If the x-component of w is 0, then R is certainly below 2.2, and if it at least 1, then the point f is so far away from the line $0w$ that the area of the triangle $0fw$ is larger than 1.

Case 2 We assume, by symmetry again, that $z-w = e_1+e_2$, and $w = (n, n-1)$ for some nonnegative integer n. Again, f is in the double disc and is outside the disc of radius $|z| - |h|$, and the argument is finished in the same way as before.

Case 3 We assume $z - w = 2e_1 + e_2$ and $w = (2n + 1, n)$ for some integer n. Again, f is in the double disc and is outside the disc of radius $|z| - |h|$ and the argument is finished similarly.

Case 4 Assume $z = w + 2e_1$, and $w = (n, 1)$ with an integer $n \geq -1$. As the area of $0zw$ is 1, K^+ is very close to the parallelogram with vertices $\pm z, \pm w$. Consequently, f is very close to z or w. But it is also in the double disc over the segment $0w$ which is impossible unless $n = -1$. In this case, however, $R \approx \sqrt{2}$, contradicting to $R > 2.2$.

Lemma 4.2 is proved. □

Next, we prove that a given z cannot serve as a moderate integer point in $\tau(K^+ \setminus K)$ for too many rotations τ.

Lemma 4.3 *Let K be such that $R \geq 3.1$, let $z \in \mathbf{Z}^2 \setminus \{0\}$, and let $J(z) = \{\tau \in S^1 : z \in \tau(K^+ \setminus K)$ is moderate$\}$. Then the measure of $J(z)$ is at most $O(\frac{\varepsilon}{R\cdot|z|})$, where* $\mathrm{vol}(K) = 4 - \varepsilon$.

Proposition 4.1 is an immediate consequence of Lemma 4.2 and Lemma 4.3. Indeed, we obtain

$$1 - P(K) \leq \sum_{z\in\mathbf{Z}^2\setminus\{0\},|z|\leq R} O\left(\frac{\varepsilon}{R\cdot|z|}\right) = O(\varepsilon)$$

as claimed by Proposition 4.1.

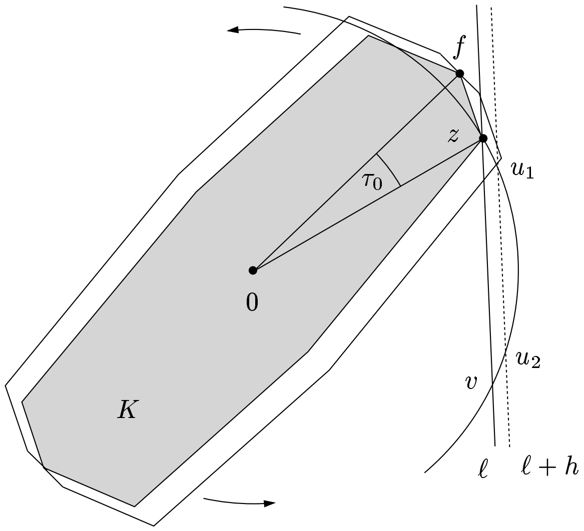


Figure 20: The proof of Lemma 4.3.

Proof of Lemma 4.3. We fix $z \in \mathbf{Z}^2 \setminus \{0\}$. First rotate K into the position where z lies on the segment $0f$, and fix this position as the starting one: $\tau = 0$. Then continue rotating in the positive (counterclockwise) direction until z first becomes moderate. This happens after a rotation by an angle τ_0; see Fig. 20. By symmetry, it suffices to bound the measure of the set $J^+(z) = \{\tau \in [0, \frac{\pi}{2}] : z \text{ is moderate for } \tau K\}$.

Let $\ell = \ell(z)$ be a line passing through z and witnessing that z is moderate for K. This ℓ intersects the circle $|z|S^1$ at points z and v. We will establish the following two conditions:

(C1) $|z - f| \leq |v - f|$.

(C2) The translated line $\ell + h$ intersects the circle $|z|S^1$.

We verify (C1). The angle $\angle z0f$ is smaller than $\frac{\pi}{2}$, so we may assume that $\angle v0f \leq \frac{\pi}{2}$ too. Since ℓ intersects the ray $0f$ beyond f, both z and v are on the same side of f. If v were closer to f than z, then $\angle vz0 < \frac{\pi}{2}$ (Fig. 21), and since we know that $\angle 0fz \leq \frac{\pi}{4}$, we would get $\angle z0f > \frac{\pi}{4}$. Then the triangle $0zf$ would have area $\frac{1}{2}|z| \cdot |f| \sin \angle z0f \geq \frac{1}{2} \cdot 3.1 \cdot \frac{1}{\sqrt{2}} > 1.09$. At the same time, this triangle is almost contained in K: by moving z by at most $|h| = \frac{\varepsilon}{R}$ we obtain a point $z' \in K$. The areas of the triangles $z0f$ and $z'0f$ differ by at most $\frac{1}{2} \cdot \frac{\varepsilon}{R} \cdot |f| < \frac{\varepsilon}{2}$, and so we would get $\mathrm{vol}(K) > 4$.

The proof of (C2) is similar. Assume that $\ell + h$ does not meet the circle $|z|S^1$ and let $w \in \ell + h$ be nearest to 0. We show that $\angle 0zv$ must be very

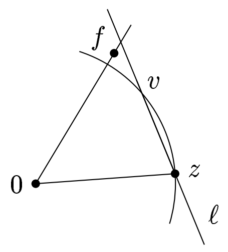


Figure 21: The argument for (C1).

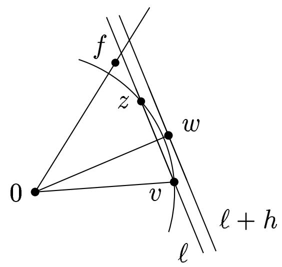


Figure 22: Bounding the angle $w0z$.

close to $\frac{\pi}{2}$; if it is the case, then the previous argument applies, showing that the area of the triangle $0zf$ would be too large. By a simple calculation (see Fig. 22), we find that $\sin \angle w0z \leq \sqrt{\varepsilon}$, and it follows that $\angle 0zv \geq \frac{\pi}{2} - \sqrt{\varepsilon}$. This proves (C2).

Let u_1 and u_2 be the intersections of $\ell + h$ with the circle $|z|S^1$, where u_1 is closer to z and u_2 is closer to v. We now claim that all the rotations $\tau \in J^+(z)$ satisfy $\tau_0 \leq \tau \leq \tau_0 + \angle z0u_1$. Indeed, $\tau_0 \leq \tau$ follows from the definition of τ_0, and for $\tau \in (\tau_0 + \angle z0u_1, \tau_0 + \angle z0u_2)$, z is too far from τK to be in $\tau(K^+ \setminus K)$. Assume now that $\tau \in [\tau_0 + \angle z0u_2, \frac{\pi}{2})$. Rotate z clockwise by angle $\tau - \tau_0$ and denote the rotated point by w. If τ were in $J^+(z)$, i.e., if z were moderate for τK, then $w \in K^+ \setminus K$, and the area of the triangle $f0w$ is too large again since $\angle f0w \geq \angle f0u_2 \geq \frac{\pi}{4}$.

Let γ denote the angle made by ℓ with the circle $|z|S^1$ at the intersection point z. We want to show that γ is bounded away from 0; see Fig. 23. This is again similar to the proof of (C1). The angle $\angle z0f$ is at least $\frac{\pi}{4} - \gamma$, and as we have calculated before, if this angle is very near to $\frac{\pi}{4}$ then the triangle $z0f$ has too large area. So we may assume $\gamma > \gamma_0$ for a positive constant γ_0.

By the above, we can estimate the measure of $J^+(z)$ by the angle $z0u_1$, and here simple estimates show that this angle is bounded by $O(\frac{\varepsilon}{|z| \cdot R\gamma})$; cf. Fig. 24. This concludes the proof of Lemma 4.3. $\square$

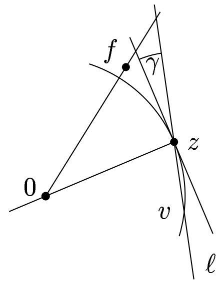


Figure 23: Bounding the angle γ.

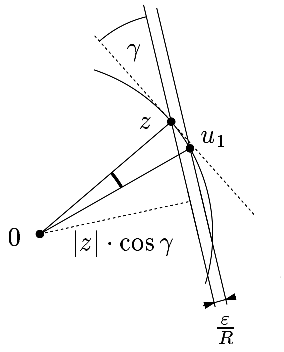


Figure 24: Bounding the angle $z0u_1$.

Note that, in the proof above, the condition $R > 3.1$ is only used in the form $|z| \cdot |f| > 3.1$. This also holds when $|z| \geq \sqrt{2}$ and $|f| = R > 2.2$. We will need this observation later in the "thick" case:

Lemma 4.4 *Let K be such that $R \geq 2.2$ and $\mathrm{vol}(K) = 4-\varepsilon$. Assume $z \in \mathbf{Z}^2$ with $|z| \geq \sqrt{2}$. Then the measure of $J(z)$ is at most $O(\frac{\varepsilon}{R\cdot|z|})$.*

4.2 Fat bodies and tiles

Proposition 4.5 *For any $\Delta > 0$ there exist $c > 0$ and $\varepsilon_1 > 0$ such that if K is a body with $\mathrm{vol}(K) = 4 - \varepsilon$ and $R \leq 3.1$, where $\varepsilon \in (0, \varepsilon_1]$, such that all rotated copies τK have Hausdorff distance at least Δ from the square Q, then $P(K) \geq 1 - c\varepsilon$. Consequently, such K cannot be minimal if ε is sufficiently small in terms of Δ.*

For the proof, we adopt the following strategy. We note that if K has a lattice-point-free position and area close to 4, then $\frac{1}{2}K$ almost tiles the plane.

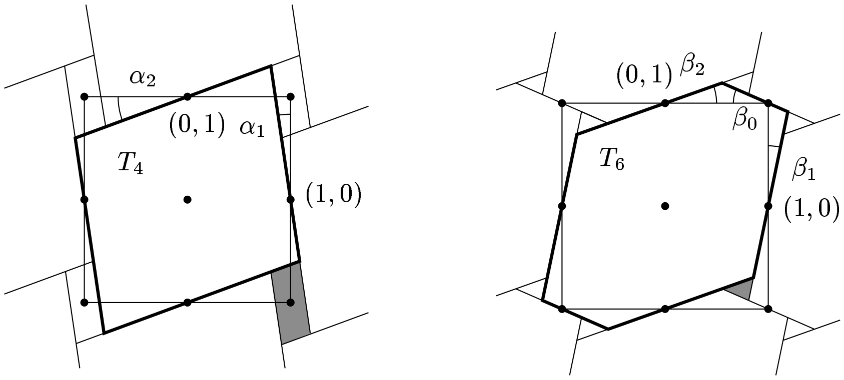


Figure 25: A basic parallelogram tile and a basic hexagonal tile.

We then discuss the possible shapes of such almost-tiles.

We define a *basic parallelogram tile* T_4 as is indicated in Fig. 25 on the left, where $0 \leq \alpha_1 \leq \alpha_2 \leq \frac{\pi}{4}$. The tiles of $T_4 + 2\mathbf{Z}^2$ have disjoint interiors, but they do not have to fill the plane completely: one of the omitted areas is indicated in the picture as a shaded parallelogram. Similarly, we define a *basic hexagonal tile* T_6 according to Fig. 25 right, with $0 \leq \beta_1, \beta_2 \leq \frac{\pi}{4}$ and $0 \leq \beta_0 \leq \frac{\pi}{2}$. A *parallelogram tile* is an image of a basic parallelogram tile under a lattice-preserving linear transform A, and similarly for a *hexagonal tile*. By a *tile* we mean a parallelogram tile or a hexagonal tile.

We recall that for a convex body $C \subset \mathbf{R}^2$, the *ε-floating body of C*, denoted by $C_{\varepsilon\text{-float}}$, is the intersection of all (closed) halfplanes h such that $\mathrm{vol}(C \cap h) \geq \mathrm{vol}(C) - \varepsilon$. We note that if C is a convex polygon with all angles bounded from below by some constant then all points of the boundary of C are at distance $O(\sqrt{\varepsilon})$ from $C_{\varepsilon\text{-float}}$.

Lemma 4.6 *Let K be a body with $\mathrm{vol}(K) = 4 - \varepsilon$ and $K \cap \mathbf{Z}^2 = \{0\}$, where $\varepsilon > 0$ is sufficiently small. Then there is a tile T with $T_{\varepsilon\text{-float}} \subseteq K \subseteq T$.*

Proof. This is similar to the usual proof of Minkowski's theorem (see e.g. [Gru95] or [PA95]). Let $K_0 = \frac{1}{2}K$ and consider the system of bodies $K_0 + \mathbf{Z}^2$; since $K \cap \mathbf{Z}^2 = \{0\}$ these bodies are pairwise disjoint. We consider the smallest $t_1 \geq 1$ such that the boundaries of the bodies in $t_1 K_0 + \mathbf{Z}^2$ first touch; say that $t_1 K_0$ touches $t_1 K_0 + z_1$. Let $K_1 = t_1 K_0$. By the symmetry and convexity of K_1, $\frac{1}{2} z_1 \in \partial K_1$. Let ℓ_1 be a line through $\frac{1}{2} z_1$ separating K_1 and $K_1 + z_1$; then K_1 is contained in the strip between ℓ_1 and $-\ell_1$. Next, enlarge K_1 to $K_1(t_2) = K_1 + [-t_2 h, t_2 h]$, where h is a unit vector parallel to ℓ_1 and $t_2 \geq 0$ is the smallest such that $K_1(t_2)$ touches some $K_1(t_2) + z_2$, $z_1 \neq z_2$. If several choices of z_2 are possible we take the one such that $\frac{1}{2} z_2$ is encountered

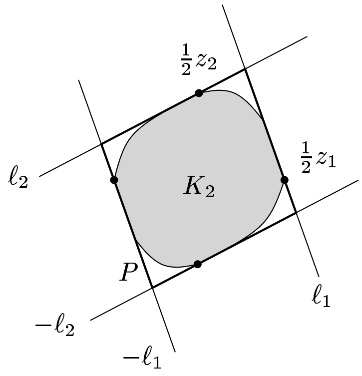


Figure 26: Proof of Lemma 4.6.

first along the boundary of the body when going from $\frac{1}{2}z_1$. Let $K_2 = K_1(t_2)$. Again $\frac{1}{2}z_2 \in \partial K_2$, and we let ℓ_2 be a line through $\frac{1}{2}z_2$ separating K_2 from $K_2 + z_2$. The body K_2 is now contained in the parallelogram P delimited by the lines $\pm\ell_1$ and $\pm\ell_2$, as in Fig. 26. If the parallelograms in $P + \mathbf{Z}^2$ do not overlap then $T = 2P$ is the desired (parallelogram) tile, as we will check later. If they do overlap then we enlarge K_2 further; in this third stage, we use the parameterization $K_2(t_3) = (1 - t_3)K_2 + t_3 P$, $t_3 \in [0, 1]$. We consider the smallest t_3 such that $K_3 = K_2(t_3)$ touches some $K_2(t_3) + z_3$, $z_3 \neq z_1, z_2$. We separate K_3 and $K_3 + z_3$ by a line ℓ_3. The lines $\pm\ell_3$ cut off P into a hexagon H, and we claim that $T = 2H$ is our tile. Indeed, if the hexagons in $H + \mathbf{Z}^2$ overlapped, then we could still enlarge K_3 so that it touches some $K_3 + z_4$, but this is impossible by a planarity argument (connect the centers of the bodies by edges through the contact points).

Since the triangle $0z_1z_2$ contains no lattice points distinct from the vertices, it has area $\frac{1}{2}$ and there is a lattice-preserving linear transform A_1 sending z_1 to e_1 and z_2 to e_2. If the parallelogram $2A_1(P)$ contains no integer point distinct from 0 in its interior then it is a basic parallelogram tile according to our definition. If it does contain such an integer point then it has to contain $(1, 1)$ or $(-1, 1)$; say that it contains $(1, 1)$. If this point is cut off by a halfplane avoiding e_1 and e_2 (and $(-1, -1)$ is cut off by the symmetric halfplane) then it is easy to check that no other integer point can remain. Thus, in this case, $T = 2H$ is a hexagonal tile.

The claim $T_{\varepsilon\text{-float}} \subseteq K$ follows from $\mathrm{vol}(K) = 4 - \varepsilon$ and $\mathrm{vol}(T) \leq 4$. $\square$

More properties of the tiles. If T had a very sharp angle then its ε-floating body would have a very large circumradius. Since we assume that K has circumradius at most 3.1, we see that T does not have any very sharp angles and ∂T is nowhere farther than $O(\sqrt{\varepsilon})$ from $T_{\varepsilon\text{-float}}$ and thus also from

K. Moreover, if a point $x \in \partial T$ is at distance at least $\delta > 0$ (independent of ε) away from the vertices of T then the distance of x to K is at most $C_\delta \varepsilon$, where C_δ is a suitable function of δ, again by simple properties of the floating body. These properties can be checked by hand (as the tiles are very simple convex polygons), or found in [Lei86].

Let A be the lattice-preserving linear map transforming a basic tile into the tile T; then Ae_1 and Ae_2 are (primitive) integer points on ∂T. The first few possible lengths of primitive integer vectors are 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$,.... We will discuss the possible values of Ae_1 and of Ae_2 and the circumradii of K in the various cases.

Since each tile has circumradius at least $\sqrt{2}$, we always have $R \geq \sqrt{2} - O(\sqrt{\varepsilon})$. Further, if $|Ae_1| > \sqrt{2}$ or $|Ae_2| > \sqrt{2}$ then $R \geq \sqrt{5} - O(\sqrt{\varepsilon})$, and if $|Ae_1| > \sqrt{5}$ or $|Ae_2| > \sqrt{5}$ then $R \geq \sqrt{10} - O(\sqrt{\varepsilon})$. The latter case is excluded by the assumption $R < 3.1$.

Thus, we know that $|Ae_1|, |Ae_2| \leq \sqrt{5}$. Since the triangle $0(Ae_1)(Ae_2)$ has area $\frac{1}{2}$, we find that necessarily $|Ae_1| = 1$ or $|Ae_2| = 1$. By symmetry, we may thus assume $Ae_2 = e_2$. Let us first consider the case where T is a parallelogram tile. By discussing the possible values of Ae_1, we arrive at the cases (a)–(e) in Fig. 27 for the parallelogram tiles.

For these parallelogram tiles, we have $\min(\alpha_1', \alpha_2') = O(\sqrt{\varepsilon})$. This is because the area left out in the tiling by T (shaded in Fig. 25) is at most ε, as $\mathrm{vol}(T) \geq \mathrm{vol}(K) = 4 - \varepsilon$.

A similar analysis yields the two possibilities (f) and (g) for hexagonal tiles shown in Fig. 28.

A case analysis. In the case (c), the tile has a too large radius.

In the cases (b), (e), and (g), we have $R \geq \sqrt{5} - O(\sqrt{\varepsilon})$ and so Lemma 4.2 applies, showing that a moderate lattice point can be made responsible for each lattice-point-free position of K, and further by Lemma 4.4, the contribution of the lattice points different from $\pm e_1, \pm e_2$ is only $O(\varepsilon)$. Moreover, troubles with $\pm e_1$ or $\pm e_2$ can occur only if the circle S^1 intersects ∂T at very small angle γ or very close to a vertex. (Here and in the sequel, by "very small" we mean "below a constant that can be made arbitrarily small by choosing ε_1 and Δ sufficiently small", but the constant must not depend on ε.) In all the cases (b), (e), (g) we thus need not worry about the intersections of ∂T with S^1 at $\pm e_2$. The other intersections of ∂T with S^1 are fine unless the following conditions hold:

- In the case (b), both the angles α_1' and α_2' are very small and so T is close to $T_0 = \mathrm{conv}\{\pm e_1, \pm(1,2)\}$.
- In the case (e), the two vertices of T closer to the origin are very close to S^1. Since $\alpha_1' = O(\sqrt{\varepsilon})$ or $\alpha_2' = O(\sqrt{\varepsilon})$, this is possible only if T is very close to a reflected copy of T_0.

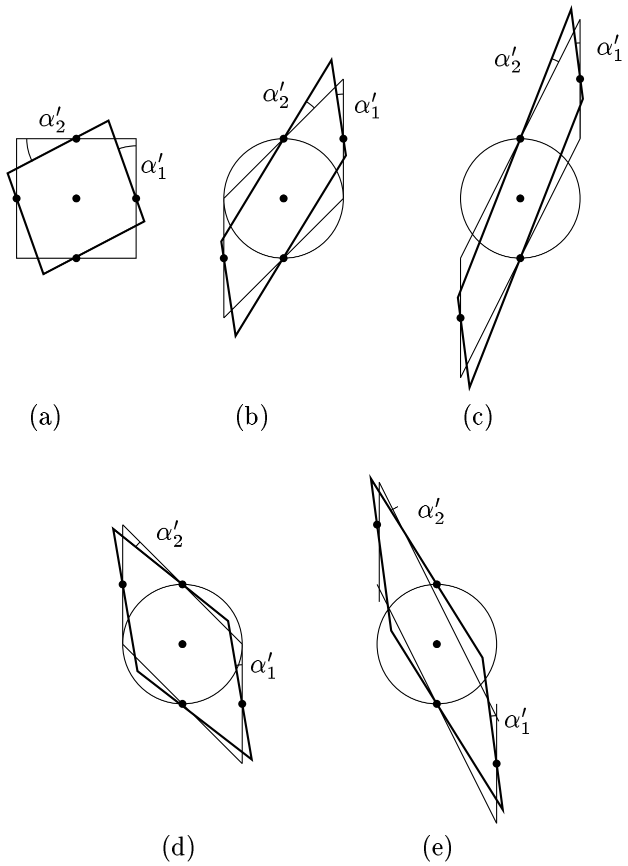


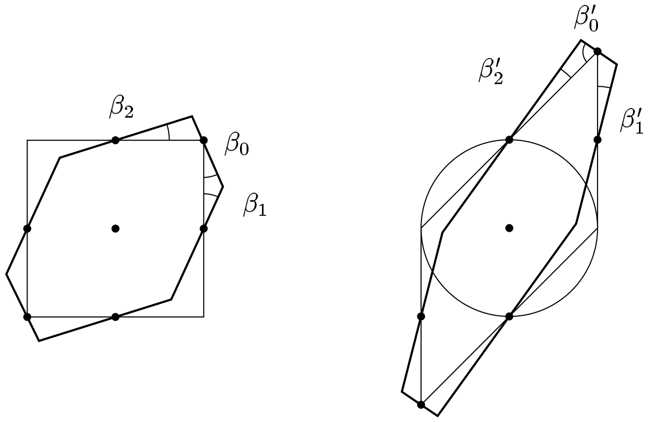
Figure 27: The possible cases of parallelogram tiles.

- In the case (g), the two vertices of T closest to the origin are very close to S^1, and then both β'_1 and β'_2 are very small and T is again close to a reflected copy of T_0.

Consider the case (b) when T is very close to T_0. Suppose that T (drawn thick) is positioned as in Fig. 29 on the left and starts rotating, together with K, counterclockwise. After angle interval $O(\varepsilon)$, e_2 enters K, and before it leaves K, $(-1, 1)$ is already in (middle picture). After $(-1, 1)$ leaves, there is an interval of length only $O(\varepsilon)$ after which e_1 enters. So $1 - P(K) = O(\varepsilon)$.

Similar arguments work for (e) and (g) and they are omitted.

We divide the case (d) into two subcases: (d1) when α'_1 is $O(\sqrt{\varepsilon})$, and (d2) when $\alpha'_2 = O(\sqrt{\varepsilon})$ (note that these cases are not symmetric).



(f) (g)

Figure 28: The possible hexagonal tiles.

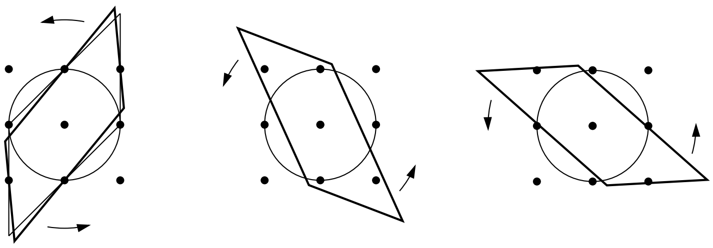


Figure 29: Rotating a tile close to T_0.

In the cases (d1), and (a), we know that two sides of T are nearly vertical and so T is very close to a parallelogram "interpolating" between T_0 and the square Q; see Fig. 30. If T starts rotating from the position drawn thick in either direction, then e_2 for clockwise direction and $(-1, 1)$ for counterclockwise direction enter K after angular interval $O(\varepsilon)$ *unless* the angle α is very small, and in that case T is very close to Q. For counterclockwise rotation, when $(-1, 1)$ is going to leave K, as in the dashed position in the drawing, $-e_1$ has already entered. We see that $1 - P(K) = O(\varepsilon)$ unless T is very close to Q. A similar analysis shows that $1 - P(K) = O(\varepsilon)$ always in the case (d2).

Finally, in the case (f), we clearly have $P(K) = O(\varepsilon)$ if neither β_1 nor β_2 are very small. Thus, suppose that β_2 is very small as in Fig. 31. Now since the area of the left-out part (shaded) must be at most 2ε, either β_1 is very small, in which case T is almost the square Q, or β_0 is very close to $\frac{\pi}{2}$, and then we are in the situation already discussed for the cases (a) and (d1).

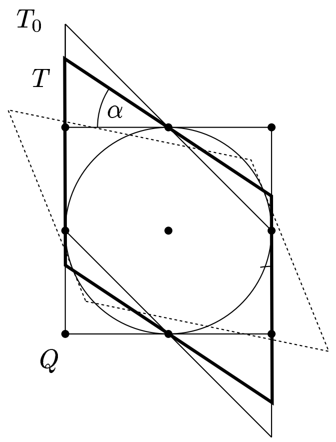


Figure 30: The cases (a) and (d1).

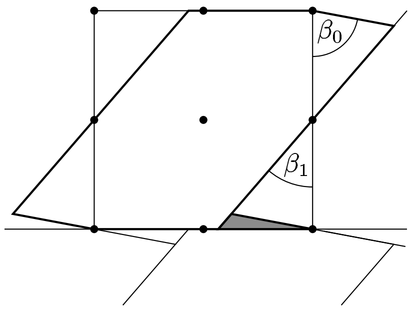


Figure 31: The case (f).

This concludes the proof of Proposition 4.5. □

4.3 Bodies very close to the square

We want to prove that $1 - P(4-\varepsilon) = O(\sqrt{\varepsilon})$ for all $\varepsilon \in [0, \varepsilon_0]$. Let K be a body at Hausdorff distance at most Δ from the square Q. By Lemma 4.6, there is a tile T with $T_{\varepsilon\text{-float}} \subseteq K \subseteq T$. Further let T^+ denote the tile T enlarged by a factor of $1 + \frac{\varepsilon}{4}$, so that $\mathrm{vol}(T^+) > 4$. Then we have $P(T^+) = 1$ by Minkowski's theorem and we can bound

$$\begin{aligned} 1 - P(K) &= (1 - P(T^+)) + (P(T^+) - P(K)) \leq P(T^+) - P(T_{\varepsilon\text{-float}}) \\ &\leq \mu(S^1 \cap (T^+ \setminus T_{\varepsilon\text{-float}})) + \mu(\sqrt{2}\, S^1 \cap (T^+ \setminus T_{\varepsilon\text{-float}})), \end{aligned}$$

where μ is the uniform probability measure on S^1.

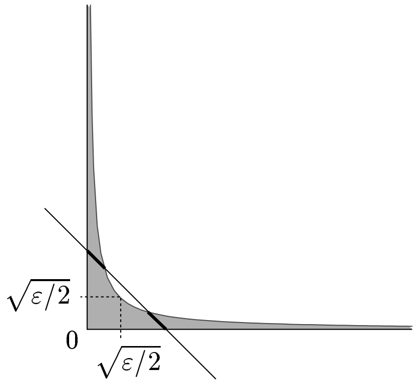


Figure 32: The floating body of a quadrant.

The types of tiles coming into consideration as T are (a), (d), and (f). The treatment of the type (f) will be slightly different.

For all these types, the intersection of $T^+ \setminus T_{\varepsilon\text{-float}}$ with S^1 is far from the corners of T^+. Near each of these intersections, the set $T^+ \setminus T_{\varepsilon\text{-float}}$ can be covered by a strip of width 2ε, say, and so $\mu(S^1 \cap (T^+ \setminus T_{\varepsilon\text{-float}}))$ is at most four times the length of the intersection of S^1 with a parallel strip of width 2ε, which is $O(\sqrt{\varepsilon})$.

We note that if T is a hexagonal tile of the type (f), then a quarter-circle of S^1 is completely contained in T. This quarter-circle alone guarantees $P(T) = 1$, and by passing from T to $T_{\varepsilon\text{-float}}$ we lose a portion of at most $O(\sqrt{\varepsilon})$ from this quarter-circle. Therefore, $1-P(K) = O(\sqrt{\varepsilon})$ follows without considering the second circle $\sqrt{2}\,S^1$ at all.

For the tile types (a) and (d), it remains to bound the length of the arcs of the second circle $\sqrt{2}\,S^1$ inside $T^+ \setminus T_{\varepsilon\text{-float}}$. We note that since T is a parallelogram tile close to the square, it has a corner with angle close to $\frac{\pi}{2}$ near $(-1,-1)$, and the second circle intersects such corner (if at all) at an angle close to $\frac{\pi}{4}$. Also, the ε-floating body of T near the second circle is almost like the ε-floating body of a full quadrant (see Fig. 32), which is bounded by the hyperbola $xy = \frac{\varepsilon}{2}$. Then $\mu(\sqrt{2}\,S^1 \cap (T^+ \setminus T_{\varepsilon\text{-float}}))$ is at most twice (say) the length of the two thick segments in Fig. 32. The latter is easily seen to be always less than $2\sqrt{\varepsilon}$. $\square$

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About Authors

Imre Bárány is affiliated with Rényi Institute of Mathematics, Hungarian Academy of Sciences, POBox 127, 1364 Budapest, Hungary, and with Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England.

Jiří Matoušek is affiliated with Department of Applied Mathematics of the Charles University and with Institute for Theoretical Computer Science (ITI), Malostranské nám. 25, 118 00 Praha 1, Czech Republic.

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