On Lipschitz mappings onto a square

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1 Introduction

The following problem was posed by Laczkovich [3]: Let $E \subseteq \mathbb{R}^d$ ($d \geq 2$) be a set with positive Lebesgue measure $\lambda^d(E) > 0$. Does there exist a Lipschitz mapping $f : \mathbb{R}^d \to Q = [0,1]^d$, such that $f(E) = Q$? Recently Preiss [4] answered this question affirmatively for $d = 2$:

Theorem 1 [4] *Let $E \subseteq \mathbb{R}^2$ be a set with $\lambda^2(E) > 0$. There exists a Lipschitz mapping f of the plane onto the square $Q = [0,1]^2$, such that $f(E) = Q$.*

In this note we give a somewhat different proof of this result[1] based on a well-known combinatorial lemma due to Erdös and Szekerés. By an additional trick, we also prove the following "absolute constant" version:

Theorem 2 *There exists a constant $c > 0$ such that for any E in the plane with Lebesgue measure $\lambda^2(E) = 1$ there exists a 1-Lipschitz mapping f of the plane onto the square $Q = [0,c]^2$, such that $f(E) = Q$.*

The value of c obtained from our proof is quite small. Clearly some improvement is possible, but it seems that our method is not suitable for obtaining the best possible value of c. It is easy to see that one cannot hope to push c arbitrarily close to 1 (e.g., consider E being a disk of unit area; its diameter $2/\sqrt{\pi}$ is smaller than the diagonal of the unit square, and so there is no 1-Lipschitz mapping onto the unit square).

Laczkovich's question for $d > 2$ remains open; a combinatorial problem related to an attempt to generalize our proof is stated in the end of section 2.

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[1]Preiss in fact proves a slightly stronger statement, namely that f can be taken such that $f(\mathbb{R}^2 \setminus E)$ is countably rectifiable (i.e. it can be covered by a countable set of Lipschitz curves). In order to keep this note technically simple, we will not prove this strengthening here, although our method also provides it with some extra care.

2 Proof of Theorem 1

The metric in $\mathbb{R}^2$, implicitly considered in Theorem 1, is the usual Euclidean metric. It appears more convenient to work with the maximum metric d_∞ defined by

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

This metric will be used throughout the rest of this paper. Clearly this modification does not affect the validity of Theorem 1.

We begin the proof by a simple lemma, which is essentially contained in [4]. Let $a > 0, w > 0$ be real numbers and $\varphi : [0, a] \to [0, a]$ be a 1-Lipschitz real function. Let Q denote the square $[0, a]^2$. We partition Q into three regions defined as follows (see Fig. 1): L (resp. S, resp. U) is the set of points $(x, y) \in Q$ with $y < \varphi(x) - w$ (resp. $\varphi(x) - w \le y \le \varphi(x) + w$, resp. $y > \varphi(x) + w$). We

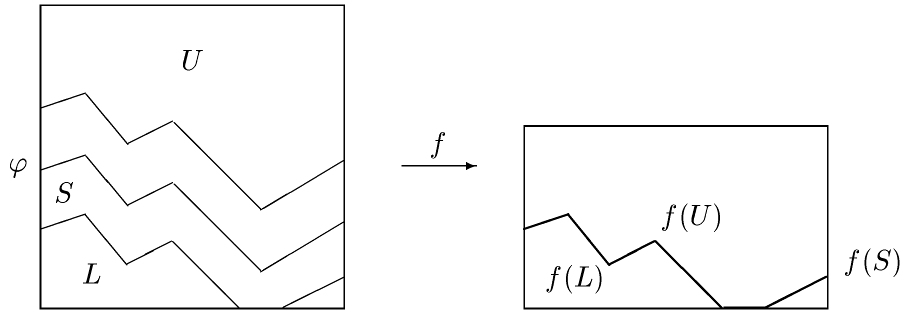


Figure 1: Contraction along φ

define a mapping $f = f_{\varphi,w} : Q \to [0, 1] \times [0, 1 - 2w]$ (called the *contraction of Q along φ by w*) as follows:

$$f(x, y) = \begin{cases} (x, \min(y, 1 - 2w)) & \text{for } (x, y) \in L \\ (x, \max(y - 2w, 0)) & \text{for } (x, y) \in U \\ (x, \max(\min(\varphi(x) - w, 1 - 2w), 0)) & \text{for } (x, y) \in S. \end{cases}$$

Lemma 3 *With a, w, φ as above, $f = f_{w,\varphi}$ is a 1-Lipschitz mapping.*

Proof: Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points in Q and $q_1 = (u_1, v_1)$ and $q_2 = (u_2, v_2)$ their f-images, resp., and suppose that $v_1 \ge v_2$. We must show $d_\infty(p_1, p_2) \ge d_\infty(q_1, q_2)$. First, if $|u_1 - u_2| \ge |v_1 - v_2|$, then $d_\infty(q_1, q_2) = |u_1 - u_2| = |x_1 - x_2| \le d_\infty(p_1, p_2)$, as f does not alter the x-coordinate. On the other hand, suppose that $|u_1 - u_2| < |v_1 - v_2| = v_1 - v_2$; then $q_1 \in f(S) \cup f(L)$ implies $q_2 \in f(L)$, as $f(S)$ is a graph of the 1-Lipschitz function $x \mapsto \max(0, \min(1 - 2w, \varphi(x) - w))$. Therefore f decreases the y-coordinate of p_1 by at least as much as the y-coordinate of p_2, and we have $d_\infty(q_1, q_2) = v_1 - v_2 \le y_1 - y_2 \le d_\infty(p_1, p_2)$. $\square$

We also note that the mapping f in the above lemma does not increase the measure of sets.

Let $E \subseteq \mathbb{R}^2$ be as in Theorem 1. By the Density Theorem for the Lebesgue measure, one can choose a square Q_0 with side a_0 and a compact subset $K \subseteq E \cap Q_0$ with $\lambda^2(K) \geq 0.99\lambda^2(Q_0)$. For notational convenience, let us assume $Q_0 = [0, a_0]^2$. Let $f_0 : \mathbb{R}^2 \to Q_0$ be a 1-Lipschitz retraction of the plane onto Q_0.

We derive Theorem 1 from the following lemma.

Lemma 4 *Let $Q = [0, a]^2$ be an axis-parallel square, let $K \subseteq Q$ be compact with $\lambda^2(K) = \lambda^2(Q)(1 - \varepsilon)$, $0 < \varepsilon \leq 0.01$. Then one can find a 1-Lipschitz mapping g of Q onto $Q' = [0, a']^2$ such that*

(i) $a' \geq a(1 - \sqrt{\varepsilon})$,

(ii) $d_\infty(p, g(p)) \leq a\sqrt{\varepsilon}$ *for any* $p \in Q$,

(iii) $\lambda^2(K') \geq \lambda^2(Q')(1 - 0.9\varepsilon)$, *where* $K' = g(K)$.

Assuming this Lemma, the proof of Theorem 1 is finished by an inductive construction as follows. Suppose that we have already constructed a square $Q_i = [0, a_i]^2$ and a 1-Lipschitz mapping f_i of $\mathbb{R}^2$ onto Q_i in such a way that $\lambda^2(K_i) \geq \lambda^2(Q_i)(1 - \varepsilon_i)$, where $K_i = f_i(K)$. We apply Lemma 4 with $Q = Q_i$, $a = a_i$, $K = K_i$, and we get a 1-Lipschitz mapping $g_i : Q_i \to Q_{i+1} = [0, a_{i+1}]^2$ with properties as in the Lemma. We set $f_{i+1} = g_i \circ f_i$ and continue by the next step of the construction.

From the Lemma we get $\varepsilon_{i+1} \leq 0.9\varepsilon_i$ and $a_{i+1} \geq a_i(1 - \sqrt{\varepsilon_i})$. We thus have $\varepsilon_i \leq 0.01(0.9)^i$, $a_i \geq a_0 \prod_{j=0}^{i-1}(1 - 0.95^j/10)$, and straightforward estimates give $a = \lim_{i\to\infty} a_i \geq 0.1a_0$.

For every point $p \in \mathbb{R}^2$, we set $f(p) = \lim_{i\to\infty} f_i(p)$. The condition (ii) of Lemma 4 guarantees that the limit exists. The mapping f is clearly 1-Lipschitz, and its image is contained in the square $Q = [0, a]^2$. The set $f(K)$ is compact and it is easily seen that it is also dense in Q, thus $f(K) = Q$. The proof of Theorem 1 is finished by rescaling f so that it maps K onto $[0, 1]^2$.

Proof of Lemma 4: Let G be the set of squares with side a/n from an $n \times n$ grid covering Q. Let $B_0 \subseteq G$ be the set of squares $s \in G$ with $\lambda^2(s \setminus K) \geq \frac{15}{16}\lambda^2(s)$. Choose n so large that the squares of B_0 contain at least 1/2 of the measure of $Q \setminus K$; this is possible by elementary measure theoretic considerations. Then $|B_0| \geq \varepsilon n^2/2$.

The required mapping g will again be constructed inductively, this time in a finite number t of steps. Let us explain the first step of the construction. Let $\varphi_0 : [0, a] \to [0, a]$ be a suitable 1-Lipschitz function; its choice is the heart of the proof. The requirement on φ_0 is that either the graph of φ_0 (i.e. the set $\{(x, \varphi_0(x));\ x \in [0, a]\}$) or the graph of φ_0 rotated by $\pi/2$ (i.e. the set $\{(\varphi_0(y), y);\ y \in [0, a]\}$) contain the centers of possibly many squares of B_0, namely at least $\sqrt{|B_0|}$ such centers. We will explain the construction assuming that the first case occurs (the graph of φ_0 contains at least $\sqrt{|B_0|}$ centers). The second case is handled symmetrically, by exchanging the role of the coordinate axes.

Let $D_0 \subseteq B_0$ be the squares whose centers are contained in the graph of φ_0, $|D_0| \geq \sqrt{|B_0|}$. We let $\bar{g}_0 : Q \to Q$ be the contraction of Q along φ_0 by a/n (see the definition above Lemma 3). We define an auxiliary mapping $h_0 : [0,a] \times [0, a(n-2)/n] \to [0, a(n-2)/n]^2$, acting as the identity on the first $n-2$ columns of the grid and contracting the last two columns to a vertical segment, and we set $g_0 = h_0 \circ \bar{g}_0$. Thus, the range of g_0 is the square $Q_1 = [0, a(n-2)/n]^2$.

It is easily checked that the mapping g_0 maps each square of the grid G into some square of G in Q_1. All squares in D_0 are contracted to pieces of Lipschitz curves.

We define $B_1 = \{s \in G;\ \exists s' \in B_0 : g_0(s') \subseteq s \text{ and } \lambda^2(g_0(s')) > 0\}$. Thus, we have $|B_1| \leq |B_0| - |D_0|$, so at least $|D_0|$ squares of B_0 are "killed" by g_0.

Similarly we define mappings $g_1, g_2, \ldots$. Having defined the mapping g_{i-1}, we put $B_i = \{s \in G;\ \exists s' \in B_{i-1} : g_{i-1}(s') \subseteq s \text{ and } \lambda^2(g_{i-1}(s')) > 0\}$ and we construct an appropriate 1-Lipschitz mapping $g_i : Q_i \to Q_{i+1}$, where $Q_i = [0, a(n-2i)/n]^2$. The mapping g_i kills at least $\sqrt{|B_i|}$ squares of B_i, in the sense that $|B_i| - |B_{i+1}| \geq \sqrt{|B_i|}$..

Let t be the first index such that $|B_0| - |B_t| \geq \varepsilon n^2/5$. For $i < t$, we thus have $|B_0| - |B_i| < \varepsilon n^2/5$ (and therefore $|B_i| > \varepsilon n^2/4$). In particular, we get $\varepsilon n^2/5 > |B_0| - |B_t| \geq \sum_{i=0}^{t-2} \sqrt{|B_i|} \geq (t-1)n\sqrt{\varepsilon}/2$, which implies that $t < \frac{2}{5} n\sqrt{\varepsilon} + 1 \leq n\sqrt{\varepsilon}/2$ (since we may assume that $n\sqrt{\varepsilon}$ is large).

By composing the mappings $g_0, \ldots, g_{t-1}$, we obtain a 1-Lipschitz mapping g of Q onto $Q' = [0, a']^2$, with $a' = a(1 - 2t/n) \geq a(1 - \sqrt{\varepsilon})$. Under this mapping, at least $\varepsilon n^2/5$ of the squares of B_0 have images of measure 0, and these squares contain at least $\frac{15}{16}(a^2/n^2)(\varepsilon n^2/5) = \frac{3}{16}\varepsilon a^2$ of the measure of $Q \setminus K$. Hence $\lambda^2(g(Q \setminus K)) \leq \frac{13}{16}\varepsilon a^2$ and $\lambda^2(g(K)) \geq \lambda^2(Q' \setminus g(Q \setminus K)) \geq a'^2 - \frac{13}{16}\varepsilon a^2$. Using the assumption $\varepsilon \leq 0.01$, we have $a' \geq a(1 - \sqrt{\varepsilon}) \geq 0.9a$, and thus $\lambda^2(g(K)) \geq a'^2(1 - \frac{10}{9} \cdot \frac{13}{16}\varepsilon) \geq a'^2(1 - 0.9\varepsilon)$ as required.

It remains to show how to choose the mappings φ_i, whose graphs or rotated graphs pass through sufficiently many center points of the squares of B_i. Call a set $D \subseteq \mathbb{R}^2$ *1-Lipschitz in the y-coordinate* (resp. *in the x-coordinate*) if $|y_1 - y_2| \leq |x_1 - x_2|$ (resp. $|x_1 - x_2| \leq |y_1 - y_2|$) for any two points $(x_1, y_1), (x_2, y_2) \in D$. It is easy to see that if D is a finite set which is 1-Lipschitz in the y-coordinate, then there is a 1-Lipschitz function whose graph contains D. Hence it suffices to establish the following

Lemma 5 *Let P be an m point set in the plane, $k = \lceil \sqrt{m} \rceil$. Then there exists either a k point subset of P 1-Lipschitz in the x-coordinate or a k point subset of P 1-Lipschitz in the y-coordinate.*

Proof: Call a set C in the plane a *nondecreasing chain* if for any two points $(x_1, y_1), (x_2, y_2) \in C$ with $x_1 \leq x_2$ we have $y_1 \leq y_2$; a *nonincreasing chain* is defined analogously with an opposite inequality for the y-coordinates. A lemma due to Erdös and Szekerés [2] can be stated as follows: Any m point set in the plane contains a k point nondecreasing chain or a k point nonincreasing chain. To see the relevance to the above lemma, rotate the coordinate system by $\pi/4$; then nondecreasing chains become precisely 1-Lipschitz subsets in the x-coordinate and nonincreasing chains become 1-Lipschitz subsets in the x-coordinate. $\square$

For a direct application of the above method for Laczkovich's problem in dimension d, one would need the following: Given an m-point set P in $\mathbb{R}^d$, find a subset S of size $cm^{1-1/d}$ ($c > 0$ a constant) which is 1-Lipschitz in the x_i-coordinate for some $i = 1, 2, \ldots, d$. Here a subset $S \subseteq P$ is C-Lipschitz in the x_i-coordinate if for any two points $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in S$ one has $|a_i - b_i| \le C \max_{i \ne j}(|a_j - b_j|)$. As noted by Gabor Tardos [5], already for $d = 3$ such a large 1-Lipschitz subset need not exist. Indeed, let $P \subset \mathbb{R}^3$ be the set $\{(i, j, i+i);\ i, j = 1, 2, \ldots \sqrt{m}\}$; it is not difficult to check that there is no subset of size exceeding $O(\sqrt{m})$ which is 1-Lipschitz in one of the coordinates. Still, it is possible that there always exists a subset of size $cm^{1-1/d}$ which is C-Lipschitz in one of the coordinates, C a constant. This problem looks interesting in its own right, although it is not completely clear whether a positive answer would solve Laczkovich's problem.

3 Proof of Theorem 2

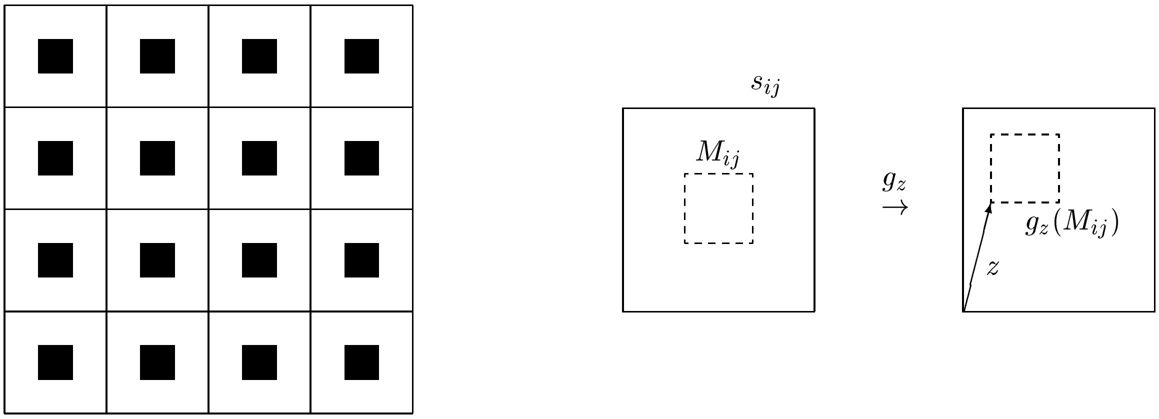
In the above proof of Theorem 1, the only part where an arbitrarily large part of the measure of E is wasted is the initial application of the Density Theorem for the choice of the square Q_0 with $\lambda^2(Q_0 \cap E) > 0.99\lambda^2(Q_0)$. In the rest of the proof, $Q_0 \cap E$ is then mapped onto a square with side at most 10 times smaller than the side of Q_0. Thus, it suffices to prove the following lemma:

Lemma 6 *For any $\varepsilon_0 \in (0, 1)$ and any $E \subset \mathbb{R}^2$ with $\lambda^2(E) = 1$, there exists a 1-Lipschitz mapping $f : \mathbb{R}^2 \to Q = [0, c]^2$ with $\lambda^2(f(E)) \ge (1 - \varepsilon_0)\lambda^2(Q)$, where $c = c(\varepsilon_0) > 0$ only depends on ε_0.*

Proof: The proof is probabilistic, using the so-called Second Moment Method, see e.g., [1].

Fix $c = \sqrt{\varepsilon_0}/7$. For simplicity, we will construct a 2-Lipschitz mapping onto $[0, c]^2$; the Lemma is then obtained by taking $c/2$ instead of c and rescaling. Choose an integer N large enough so that some axis-parallel square S with side $3cN$ contains at least $9/10$ of the measure of E. Let G be an $N \times N$ grid consisting of $3c \times 3c$ axis-parallel squares; the squares of G are denoted by s_{ij}, $i = 1, 2, \ldots N$ denoting the row index and $j = 1, 2, \ldots, N$ the column index. Let M_{ij} denote the middle $c \times c$ square in s_{ij}, and set $M = \bigcup_{i,j=1}^{N} M_{ij}$. Since 9 translational copies of the set M cover the square S, it is possible to place the grid G in such a way that $\lambda^2(M \cap E) \ge 1/10$; we assume that such a placement has been fixed.

Consider a grid square s_{ij}. Let $z = z_{ij} \in [0, 2c]^2$ be a vector. We define a mapping $g_z : s_{ij} \to s_{ij}$ as follows. We require that g_z is the identity map on the boundary ∂s_{ij} of s_{ij}, and on M_{ij}, g_z acts as the translation by the vector $z - (c, c)$ (see Fig. 2). With respect to the d_∞ metric, g_z is a 2-Lipschitz mapping on $\partial s_{ij} \cup M_{ij}$. We extend it to a 2-Lipschitz mapping $s_{ij} \to s_{ij}$ (this is possible since for any metric space Y and any $X \subseteq Y$, a C-Lipschitz mapping from X to the plane with the d_∞ metric can be extended to a C-Lipschitz mapping defined on Y, see e.g., [6]).



(a) (b)

Figure 2: (a) The grid G and the set M; (b) the mapping g_z.

For later use, we observe that if z is randomly chosen from the uniform probability distribution on $[0, 2c]^2$, then for any fixed $x \in M_{ij}$, we have

$$\mathrm{Prob}(x \in g_z(E \cap M_{ij})) \geq \frac{\lambda^2(E \cap M_{ij})}{4c^2} \tag{1}$$

(since for each $y \in E \cap M_{ij}$, $g_{x-y+(c,c)}$ sends y to x).

Consider $Z = \{(z_{ij};\ i,j = 1,2,\ldots,N);\ z_{ij} \in [0,2c]^2\}$ with the product probability measure, and let $\mathbf{z} \in Z$ be a random element. Let $\bar{g}_{\mathbf{z}} : G \to G$ be the 2-Lipschitz mapping whose restriction on each s_{ij} is $g_{z_{ij}}$. Further we define a 1-Lipschitz mapping $f_1 : G \to [0, 3c]^2$ corresponding to a "folding" of the grid G (as if G were a piece of paper and we wanted to fold it to one little square). Formally, if $p \in s_{ij}$ is a point in G with displacement (x, y) from the lower left corner of s_{ij}, we set

$$f_1(p) = \begin{cases} (x,y) & \text{for } i,j \text{ odd,} \\ (x, 3c-y) & \text{for } i \text{ odd, } j \text{ even,} \\ (3c-x, y) & \text{for } i \text{ even, } j \text{ odd,} \\ (3c-x, 3c-y) & \text{for } i,j \text{ even.} \end{cases}$$

Finally, we let f_0 be the retraction of the plane onto the grid G, and $f_2 : [0,3c]^2 \to Q$ be the mapping translating the middle $c \times c$ square of $[0,3c]^2$ onto $Q = [0,c]^2$ and contracting the rest of $[0,3c]^2$ onto the boundary of Q. Both f_0, f_2 are 1-Lipschitz. We put $f_{\mathbf{z}} = f_2 \circ f_1 \circ \bar{g}_{\mathbf{z}} \circ f_0$. This is a 2-Lipschitz map.

Fix any point $x \in [0,c]^2$, and let X_{ij} be the 0/1 indicator variable for the event $x \in f_{\mathbf{z}}(E \cap M_{ij})$ (this is a random variable depending on the choice of $\mathbf{z}$), and set $X = \sum_{i,j=1}^{N} X_{ij}$. From (1), we get for the expectation of each X_{ij}

$$\mathrm{E}\, X_{ij} \geq \frac{1}{4c^2} \lambda^2(E \cap M_{ij}),$$

and so

$$\mathrm{E}\,X = \sum_{i,j=1}^{N} \mathrm{E}\,X_{ij} \geq \frac{1}{40c^2} > \frac{1}{\varepsilon_0}\,.$$

As the variables X_{ij} are independent, we have

$$\mathrm{Var}\,X = \sum_{i,j=1}^{N} \mathrm{Var}\,X_{ij} \leq \sum_{i,j=1}^{N} \mathrm{E}\,X_{ij}^2 = \sum_{i,j} \mathrm{E}\,X_{ij} = \mathrm{E}\,X\,.$$

From Chebyshev Inequality we thus get

$$\mathrm{Prob}(X=0) \leq \mathrm{Prob}(|X - \mathrm{E}\,X| \geq \mathrm{E}\,X) \leq \frac{\mathrm{Var}\,X}{(\mathrm{E}\,X)^2} \leq \frac{1}{\mathrm{E}\,X} < \varepsilon_0\,.$$

The event $X \neq 0$ means $x \in f_{\mathbf{z}}(M \cap E)$, where x is our fixed point of Q. Considering all x's simultaneously and using Fubini's Theorem, we get that there exists a $\mathbf{z}_0 \in Z$ such that $\lambda^2(f_{\mathbf{z}_0}(E \cap M)) \geq (1-\varepsilon_0)\lambda^2(Q)$. This concludes the proof of Lemma 6. $\square$

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