Topological lower bounds for the chromatic number: A hierarchy

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Abstract

Kneser's conjecture, first proved by Lovász in 1978, states that the graph with all k-element subsets of $\{1, 2, \ldots, n\}$ as vertices and all pairs of disjoint sets as edges has chromatic number $n - 2k + 2$. Several other proofs have been published (by Bárány, Schrijver, Dol'nikov, Sarkaria, Kříž, Greene, and others), all of them based on the Borsuk–Ulam theorem from algebraic topology, but otherwise quite different. Each can be extended to yield some lower bound on the chromatic number of an arbitrary graph.

We show that these bounds are almost linearly ordered by strength, the strongest one being essentially Lovász' original bound in terms of a neighborhood complex. We also present and compare various definitions of a *box complex* of a graph (developing ideas of Alon, Frankl, and Lovász and of Kříž). A suitable box complex is equivalent to Lovász' complex, but the construction is simpler and functorial, mapping graphs with homomorphisms to $\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps.

1 Introduction

The Kneser conjecture [12] is concerned with the chromatic numbers of certain graphs. These graphs are quite special, but the proof methods that are known for it, starting with Lovász's [15] break-through, extend beyond the original examples: Each of them yields, explicitly or implicitly, a lower bound for the chromatic number of any graph (although these bounds are rather weak for some classes of graphs).

Although some proofs of Kneser's conjecture can be combinatorialized (Matoušek [17], Ziegler [25]), the underlying ideas of all the known proofs are topological, based on the Borsuk–Ulam theorem and its extensions. The proof methods are diverse, though, and at first sight, they look almost unrelated.

We show that, surprisingly, the topological lower bounds for the chromatic number resulting from known proofs fall neatly into a hierarchy, which is essentially linearly ordered. So, we show that "(the index version of) the Lovász' bound is stronger than the (generalized) Sarkaria bound, which is stronger than the (generalized) Bárány bound, and also stronger than the Dol'nikov–Kříž bound." (This, of course, should not indicate any comparison of the usefulness of the methods or of the interest of the papers. The hierarchy concerns only the bounds that, *in principle*, can be obtained by straightforward generalizations of these approaches, and it does not say anything about the feasibility of actually obtaining these bounds.)

The Lovász bound, which is currently the strongest, can be equivalently expressed using *box complexes*, which seem to have some technical advantages. The *idea* of that proof, as was beautifully summarized by Lovász (personal communication), is simple and striking: If c is a proper m-coloring of a graph $G = (V, E)$, then this implies that whenever two disjoint sets A', A'' of vertices are completely interconnected in G (that is, there is a complete bipartite subgraph with shores A' and A''), two disjoint sets $c(A')$ and $c(A'')$ of colors are assigned to them. Furthermore, if $B' \supseteq A'$ and $B'' \supseteq A''$ give a larger complete bipartite subgraph, they receive larger color sets $c(B') \supseteq c(A')$ and $c(B'') \supseteq c(A'')$. Third, if we interchange A' and A'', the color sets are interchanged as well. *Thus*, if we suitably define a "complex of complete bipartite subgraphs of G" (called a *box complex*), then every m-coloring yields a $\mathbb{Z}_2$-map (antipodal map) from the box complex of G to the box complex of K_m. Hence

$$\operatorname{ind}(\text{box complex of } G) \;\leq\; \operatorname{ind}(\text{box complex of } K_m) \;\sim\; \chi(G),$$

where $\operatorname{ind} X$ is the smallest k such that there is a $\mathbb{Z}_2$-map from X to the k-sphere S^k (with the usual antipodal action).

Interestingly, this idea has a number of different implementations: There are several distinct possibilities to define "box complexes." The different box complexes have different ground sets, they are of different sizes (the numbers of vertices/faces differ on an exponential scale!), and some of them may be considerably easier to use than others.

Our current favourite is the box complex $\mathsf{B}(G)$, defined in Section 3 below. It has a small vertex set (the disjoint union of two copies of $V(G)$), and it yields the strongest bounds available. However, we invite the reader to survey the panorama and to make his/her own choices—several more versions of box complexes are discussed in Section 5.

The hierarchy of the bounds and the relations among the various box complexes, once revealed, are not so hard to prove. We regard noticing them as the main contribution of this paper.

2 Preliminaries

Here we recall some general notions, facts, and notation needed for the statement of our results. Readers with a basic knowledge of the area may perhaps want to skip this part and refer to it as needed during further reading. For more background and a survey of topological methods in combinatorics we refer to Björner [5], Živaljević [26], or the forthcoming textbook treatment in Matoušek [16].

Graphs. The vertex set of a graph G is written as $V(G)$, and the edge set as $E(G)$. We suppose that all graphs are finite, simple, and undirected. In order to avoid some trivial special cases, we also assume that the considered graphs have no isolated vertices.

A *homomorphism* of a graph G into a graph H is a mapping $f\colon V(G) \to V(H)$ that preserves edges; that is, $\{f(u), f(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$. For our purposes, it is convenient to regard a (proper) coloring of G by m colors as a homomorphism of G into the complete graph K_m. The chromatic number of G is denoted by $\chi(G)$.

We regard a *bipartite graph* as a triple (V', V'', E), where $V', V'' \subseteq V$ are disjoint and $E \subseteq \{\{v', v''\} : v' \in V', v'' \in V''\}$. For a bipartite graph we assume that the bipartition is fixed, and we also distinguish (V', V'', E) from (V'', V', E). Since we will be concerned with various colorings of graphs, we call the parts V' and V'' of the bipartition the *shores*, rather than the more common "color classes." If A', A'' are disjoint subsets of the vertex set of some graph G, we write $G[A', A'']$ for the *bipartite* subgraph with shores A' and A'' induced by G. (Note that this is not necessarily an induced subgraph of G, since only edges between distinct shores are included.)

Kneser graphs. Let X be a finite set and $\mathcal{F} \subseteq 2^X$ a system of subsets of X. The *Kneser graph* $\mathrm{KG}(\mathcal{F})$ has vertex set $\mathcal{F}$, and the edges are all pairs of disjoint sets in $\mathcal{F}$. For notational convenience, we assume that $X = [n] := \{1, 2, \ldots, n\}$, unless stated otherwise. Kneser's conjecture can be succinctly stated as $\chi(\mathrm{KG}(\binom{[n]}{k})) = n - 2k + 2$ for $n \geq 2k > 0$, where $\binom{[n]}{k}$ denotes the family of all k-element subsets of $[n]$.

It is easy to see that *every* (finite) graph $G = (V, E)$ can be represented as a Kneser graph of some set system. A simple and natural representation is this: Let $\overline{E} := \binom{V}{2} \setminus E$ denote the set of non-edges of G, and for every $v \in V$, let us set $F_v := \{\overline{e} \in \overline{E} : v \in \overline{e}\}$. The Kneser graph of $\{F_v : v \in V\}$ is isomorphic to G; the only problem is that the sets F_v need not be all distinct (for example, for $G = K_n$, we have $F_v = \emptyset$ for all v). To remedy this, one can define $F'_v := F_v \cup \{v\}$, obtaining distinct sets. For a more economical representation, we can let $\mathcal{C}$ be a covering of $\overline{E}$ by cliques (each $C \in \mathcal{C}$ is a complete subgraph of $(V, \overline{E})$ and each edge of $\overline{E}$ is contained in some $C \in \mathcal{C}$). For $v \in V$, we then define $F''_v := \{C \in \mathcal{C} : v \in C\}$; this is a potentially much smaller Kneser representation. The problem of finding a Kneser representation with the smallest ground set, i.e., the smallest $\mathcal{C}$, is the minimum clique cover for the complement of G, and hence NP-complete and hard to approximate; see, e.g., Ausiello et al. [2].

Simplicial complexes. We use letters like $\mathsf{K}, \mathsf{L}, \ldots$ to denote simplicial complexes. (See, e.g., [19], [5], [16] for more background). We consider only finite simplicial complexes, so a simplicial complex K is a nonempty hereditary set system (i.e., $S \in \mathsf{K}$ and $S' \subset S$ implies $S' \in \mathsf{K}$); in particular, $\emptyset \in \mathsf{K}$. For example, $2^{[n]}$ is the $(n-1)$-dimensional simplex considered as a simplicial complex. We let $V(\mathsf{K})$ denote the vertex set of K, and $\|K\|$ denotes the polyhedron of K (but sometimes we write just K for the polyhedron too, when it is clear that we mean a topological space). The *dimension* of the complex K is $\dim \mathsf{K} := \max\{|S| - 1 : S \in \mathsf{K}\}$. A *simplicial map* of a simplicial complex K to a simplicial complex L is a map $f \colon V(\mathsf{K}) \to V(\mathsf{L})$ such that $f(S) \in \mathsf{L}$ for all $S \in \mathsf{K}$.

For a partially ordered set $(X, \preceq)$, the *order complex* $\Delta(X, \preceq)$ has X as the vertex set and all chains as simplices; that is, a simplex has the form $\{x_1, x_2, \ldots, x_k\} \subseteq X$ with $x_1 \prec x_2 \prec \cdots \prec x_k$. In particular, if $\mathcal{F}$ is a set system, we write $\Delta\mathcal{F}$ for $\Delta(\mathcal{F} \setminus \{\emptyset\}, \subseteq)$. If K is a simplicial complex, then $\Delta\mathsf{K}$ is the *first barycentric subdivision* of K, also denoted by $\mathrm{sd}\,\mathsf{K}$ (the empty simplex $\emptyset$ is not a vertex of the barycentric subdivision, and this is the reason for removing $\emptyset$ in the definition of $\Delta\mathcal{F}$).

The (twofold) *deleted join* of K, denoted by K^{*2}_{Δ}, has vertex set $V(K) \times [2]$ (two copies of $V(\mathsf{K})$) and the simplices are $\{S_1 \uplus S_2 : S_1, S_2 \in \mathsf{K}, S_1 \cap S_2 = \emptyset\}$, where we use the shorthand $S_1 \uplus S_2 := (S_1 \times \{1\}) \cup (S_2 \times \{2\})$.

$\mathbb{Z}_2$-spaces and $\mathbb{Z}_2$-index. A *$\mathbb{Z}_2$-space* (also called *antipodality space* in the literature) is a pair (T, ν), where T is a topological space and $\nu \colon T \to T$, called the *$\mathbb{Z}_2$-action*, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{id}_T$. If (T_1, ν_1) and (T_2, ν_2) are $\mathbb{Z}_2$-spaces, a $\mathbb{Z}_2$-map between them is a continuous mapping $f \colon T_1 \to T_2$ such that $f \circ \nu_1 = \nu_2 \circ f$. The sphere S^n is considered as a $\mathbb{Z}_2$-space with the antipodal homeomorphism $x \mapsto -x$. Following Živaljević

[26], we define the $\mathbb{Z}_2$-*index* of a $\mathbb{Z}_2$-space (T, ν) by

$$\operatorname{ind}(T, \nu) \;:=\; \min\{n \geq 0 : \text{there is a } \mathbb{Z}_2\text{-map } (T, \nu) \to S^n\} \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

(the $\mathbb{Z}_2$-action ν is omitted from the notation if it is clear from context). If $\operatorname{ind}(T_1, \nu_1) > \operatorname{ind}(T_2, \nu_2)$, then there is no $\mathbb{Z}_2$-map $T_1 \to T_2$. The Borsuk–Ulam theorem can be re-stated as $\operatorname{ind}(S^n) = n$.

A *simplicial* $\mathbb{Z}_2$*-complex* is a simplicial complex K with a simplicial map ν of K into itself such that (the canonical affine extension of) ν is a $\mathbb{Z}_2$-action on $\|\mathsf{K}\|$. For the deleted join K^{*2}_{Δ}, we have the canonical $\mathbb{Z}_2$-action given by "swapping the two copies of $V(\mathsf{K})$," formally $(v, 1) \mapsto (v, 2)$ and $(v, 2) \mapsto (v, 1)$.

For any simplicial $\mathbb{Z}_2$-complex K whose $\mathbb{Z}_2$-action is free (that is, has no fixed point), we have

$$\dim \mathsf{K} \geq \operatorname{ind} \mathsf{K} \geq 1 + \mathbb{Z}_2\text{-acyclicity}(\mathsf{K}) \geq 1 + \text{connectivity}(\mathsf{K}).$$

Here the first inequality needs freeness (in fact, $\operatorname{ind} \mathsf{K} = \infty$ if the $\mathbb{Z}_2$-action has a fixed point). The second inequality is a homological version of the Borsuk–Ulam theorem; see Walker [23]. The parameter connectivity(K) denotes the smallest k such that there exists a continuous map $S^{k+1} \to \|\mathsf{K}\|$ that is not nullhomotopic, while the acyclicity parameter is defined by

$$\mathbb{Z}_2\text{-acyclicity}(\mathsf{K}) \;:=\; \max\{k : \widetilde{\mathrm{H}}_i(\mathsf{K}, \mathbb{Z}_2) = 0 \text{ for all } i \leq k\}.$$

The $\mathbb{Z}_2$-acyclicity is of interest in this context, since it is effectively computable, both theoretically [19, §11] and practically (for not too large complexes; see [10]), while the $\mathbb{Z}_2$-index and the connectivity are in general harder to determine.

We recall that two topological spaces X and Y are *homotopy equivalent* if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. For $\mathbb{Z}_2$-spaces, $\mathbb{Z}_2$-homotopy equivalence is defined analogously, but we require that f, g, as well as all maps in the two homotopies be $\mathbb{Z}_2$-maps.

3 Proof methods for Kneser's conjecture

The box complex $\mathsf{B}(G)$. For a graph G and any subset $A \subseteq V(G)$, let

$$\mathrm{CN}(A) \;:=\; \{v \in V(G) : \{a, v\} \in E(G) \text{ for all } a \in A\} \;\subseteq\; V \setminus A$$

be the set of all *common neighbors* of A.

We define the *box complex* $\mathsf{B}(G)$ of a graph G as the simplicial complex with vertex set $X = V(G) \times [2]$ (i. e., two disjoint copies of $V(G)$), with simplices given by

$$\begin{aligned} \mathsf{B}(G) \;:=\; \big\{A' \uplus A'' :\; & A', A'' \subseteq V(G),\; A' \cap A'' = \emptyset, \\ & G[A', A''] \text{ is complete},\; \mathrm{CN}(A'), \mathrm{CN}(A'') \neq \emptyset\big\}. \end{aligned}$$

(We recall the notation $A' \uplus A'' = (A' \times \{1\}) \cup (A'' \times \{2\})$.) So the simplices of $\mathsf{B}(G)$ correspond to complete bipartite subgraphs in G. We admit A' or A'' empty, but then it is required that all vertices of the other shore have a common neighbor (if both A' and A'' are nonempty, the condition $\mathrm{CN}(A'), \mathrm{CN}(A'') \neq \emptyset$ is superfluous).

However, if the extra condition on "having a common neighbor" is deleted, then we get a *different* box complex

$$\mathsf{B}_0(G) \ := \ \big\{A' \uplus A'' : A', A'' \subseteq V(G),\ A' \cap A'' = \emptyset,\ G[A', A''] \text{ is complete}\big\}$$

that contains $\mathsf{B}(G)$, and which will also play a role in the following.

A canonical simplicial $\mathbb{Z}_2$-action on $\mathsf{B}(G)$ is given by interchanging the two copies of $V(G)$; that is, $(v,1) \mapsto (v,2)$ and $(v,2) \mapsto (v,1)$, for $v \in V(G)$. This makes $\mathsf{B}(G)$ into a $\mathbb{Z}_2$-space.

If $f\colon V(G) \to V(G)$ is a graph homomorphism, we associate to it a map $\mathsf{B}(f)\colon V(\mathsf{B}(G)) \to V(\mathsf{B}(H))$ in the obvious way: $\mathsf{B}(f)(v,j) := (f(v), j)$ for $v \in V(G)$, $j \in [2]$. It is easily verified that $\mathsf{B}(f)$ is a simplicial $\mathbb{Z}_2$-map of $\mathsf{B}(G)$ into $\mathsf{B}(H)$. Moreover, the construction respects the composition of maps, and so $\mathsf{B}(.)$ can be regarded as a functor from the category of graphs with homomorphisms into the category of $\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps.

It is not hard to show that $\mathsf{B}(K_m)$ is $\mathbb{Z}_2$-homotopy equivalent to S^{m-2}, and $\operatorname{ind}\mathsf{B}(K_m) = m-2$ (see Section 5). Since an m-coloring of G can be regarded as a homomorphism of G into K_m, it induces a $\mathbb{Z}_2$-map of $\mathsf{B}(G)$ into S^{m-2}, and we obtain

$$\chi(G) \geq \operatorname{ind}\mathsf{B}(G) + 2. \tag{1}$$

The box complex $\mathsf{B}(G)$ is a variation of ideas from Alon, Frankl, and Lovász [1] and Kříž [13].

Neighborhood complexes and the Lovász bound. Lovász [15] defined the *neighborhood complex* as $\mathsf{N}(G) := \{S \subseteq V(G) : \mathrm{CN}(S) \neq \emptyset\}$, and he proved that one always has $\chi(G) \geq 3 + \operatorname{connectivity}(\mathsf{N}(G))$. His proof uses another simplicial complex $\mathsf{L}(G)$, which can be defined as the order complex of the system of all "closed sets" in $\mathsf{N}(G)$:

$$\mathsf{L}(G) \ := \ \Delta\{A \subset V(G) : \mathrm{CN}(\mathrm{CN}(A)) = A\}.$$

Thus, the vertices of $\mathsf{L}(G)$ are shores of inclusion-maximal complete bipartite subgraphs of G. Unlike $\mathsf{N}(G)$, this $\mathsf{L}(G)$ is a simplicial $\mathbb{Z}_2$-complex, with the $\mathbb{Z}_2$-action given by $A \mapsto \mathrm{CN}(A)$, and a slight modification of Lovász' proof actually yields the lower bound

$$\chi(G) \geq \operatorname{ind}\mathsf{L}(G) + 2.$$

As shown in [15], $\mathsf{L}(G)$ is a strong deformation retract of $\mathsf{N}(G)$. A version of Lovász' bound, formulated in terms of the classifying map of the $\mathbb{Z}_2$-bundle associated with $\mathsf{L}(G)$, was published by Milgram and Zvengrowski [18]. Their formulation can be used as a tool for bounding $\operatorname{ind}\mathsf{L}(G)$ from below.

In Section 5 we will show that $\operatorname{ind}\mathsf{L}(G) = \operatorname{ind}\mathsf{B}(G)$. So while $\mathsf{B}(G)$ and $\mathsf{L}(G)$ provide the same lower bound, the functoriality of $\mathsf{B}(.)$ (which was probably known to experts, but as far as we know, hasn't appeared in print) is a significant advantage. Walker [22] shows how a homomorphism induces a $\mathbb{Z}_2$-map for the $\mathsf{L}(.)$ complexes, but the construction is more complicated and not canonical.

The subsequent lower bounds are formulated for Kneser graphs, in terms of the defining set system $\mathcal{F}$. The next definition is crucial in their formulation. With a set system $\mathcal{F}$, we associate the following simplicial complex $\mathsf{K} = \mathsf{K}(\mathcal{F})$: The vertex set of K is X, the ground set of $\mathcal{F}$, and

$$\mathsf{K}(\mathcal{F}) \ := \ \{S \subseteq X : F \not\subseteq S \text{ for all } F \in \mathcal{F}\}.$$

Thus $\mathcal{F}$ is the family of "minimal nonfaces" of K (plus possibly additional nonfaces), while K is the complex of "$\mathcal{F}$-free sets."

The Sarkaria bound. From Sarkaria's proof of Kneser's conjecture [20], the following general bound can be deduced ($\mathcal{F}$ is assumed to have the ground set $[n]$):

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \operatorname{ind} \Delta\Big((2^{[n]})^{*2}_{\Delta} \setminus \mathsf{K}^{*2}_{\Delta}\Big) + 1.$$

As it turns out, the complex on the right-hand side is just another version of a box complex of $\mathrm{KG}(\mathcal{F})$. Sarkaria, in the concrete cases he deals with, then proceeds to estimate the index of that complex using an elegant trick with joins ("Sarkaria's inequality"; see [26] or [16]), which in general leads to

$$\chi(\mathrm{KG}(\mathcal{F})) \geq n - 1 - \operatorname{ind} \mathsf{K}^{*2}_{\Delta}. \tag{2}$$

We call the right-hand side of this inequality the Sarkaria bound. It is not explicitly stated in this way in Sarkaria's papers, and so perhaps "generalized Sarkaria bound" would be more precise, but repeating the adjective "generalized" at every occasion seems annoying.

Bárány's proof from [4] yields a lower bound that can generally be phrased as follows. *Suppose that for some $d \geq 1$, the ground set X of $\mathcal{F}$ can be placed into the sphere S^d in such a way that for every open hemisphere H there exists a set $F \in \mathcal{F}$ with $F \subseteq X \cap H$. Then $\chi(\mathrm{KG}(\mathcal{F})) \geq d{+}2$.* (Kneser's conjecture is obtained from this using Gale's lemma, stating that, for every $d, k \geq 1$, one can place $2k{+}d$ points on S^d so that every open hemisphere contains at least k points.) For the purposes of comparing the bound with the other bounds, we will rephrase it using the Gale transform; see Section 6. The result can be expressed as follows: *Suppose that K is a subcomplex of the boundary complex of an $(n{-}d)$-dimensional convex polytope P (under a suitable identification of the vertices of K with the vertices of P). Then $\chi(\mathrm{KG}(\mathcal{F})) \geq d$.* We will refer to a number d as in this statement as the Bárány bound; a comment similar to the one for the Sarkaria bound applies here as well.

From this form it is not hard to show that the Sarkaria bound is always at least as strong as the Bárány bound (but, of course, the index in (2) might be difficult to evaluate).

The Dol'nikov–Kříž bound is a purely combinatorial lower estimate for $\chi(\mathrm{KG}(\mathcal{F}))$. For a set system $\mathcal{F}$, let the 2-*colorability defect* $\mathrm{cd}_2(\mathcal{F})$ (called the *width* in [13]) be the minimum size of a subset $Y \subseteq X$ such that the system of the sets of $\mathcal{F}$ that contain no points of Y is 2-colorable. In other words, we want to color each point of X red, blue, or white in such a way that no set of $\mathcal{F}$ is completely red or completely blue (it may be completely white), and $\mathrm{cd}_2(\mathcal{F})$ is the minimum number of white points required for such a coloring. The following bound was derived by Dol'nikov [8, 9] by a geometric argument from the Borsuk–Ulam theorem, and independently (and as a part of a more general result) by Kříž [13, 14], via certain box complexes:

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F}). \tag{3}$$

(Since it is easily seen that $\mathrm{cd}_2(\binom{[n]}{k}) = n - 2k + 2$, Kneser's conjecture follows.) A very short and elegant geometric reduction to a suitable version of the Borsuk–Ulam theorem follows immediately from the recent work of Greene [11], which currently provides the shortest self-contained proof of the Kneser conjecture.

The inequality (3) is also an immediate consequence of (2). Indeed, estimating the $\mathbb{Z}_2$-index by the dimension, (2) leads to $\chi(G) \geq n - 1 - \dim \mathsf{K}^{*2}_{\Delta}$, and some unwrapping of definitions reveals that, surprisingly, the latter quantity is exactly $\mathrm{cd}_2(\mathcal{F})$.

4 The hierarchy

In the following theorem, we summarize and compare all the considered lower bounds for $\chi(G)$.

Theorem 1 (The Hierarchy Theorem). *Let $G = (V, E) = \mathrm{KG}(\mathcal{F})$ be a finite (Kneser) graph with no isolated vertices, where $\mathcal{F} \subseteq 2^{[n]}$, and let $\mathsf{K} = \mathsf{K}(\mathcal{F}) = \{S \subseteq [n] : F \not\subseteq S$ for all $F \in \mathcal{F}\}$. Then we have the following chain* of inequalities and equalities:*

$$
\begin{array}{llll}
\chi(G) & \overset{\text{(H1)}}{\geq} \ \operatorname{ind} \mathsf{B}(G) + 2 \ \overset{\text{(H2)}}{=} \ \operatorname{ind} \mathsf{L}(G) + 2 & & \textit{“the Lovász bound”} \\
 & \overset{\text{(H3)}}{\geq} \ \operatorname{ind} \mathsf{B}_0(G) + 1 \ \overset{\text{(H4)}}{=} \ \operatorname{ind} \Delta\Big((2^{[n]})^{*2}_{\Delta} \setminus \mathsf{K}^{*2}_{\Delta}\Big) + 1 & & \\
 & \overset{\text{(H5)}}{\geq} \ n - 1 - \operatorname{ind} \mathsf{K}^{*2}_{\Delta} & & \textit{“the Sarkaria bound”} \\
 & \left\{ \begin{array}{ll} \overset{\text{(H6)}}{\geq} & d \quad \textit{if } \mathsf{K} \subseteq \partial P \textit{ for an } (n{-}d)\textit{-polytope } P \\ \overset{\text{(H7)}}{\geq} & n - 1 - \dim \mathsf{K}^{*2}_{\Delta} \ = \ \mathrm{cd}_2(\mathcal{F}) \end{array} \right. & & \begin{array}{l} \textit{“the Bárány bound”} \\ \textit{“the Dol'nikov–Kříž bound.”} \end{array}
\end{array}
$$

We have already proved (H1) (which is identical to (1)), as well as (H7). The inequality (H5) was essentially proved by Sarkaria ([16] contains a detailed proof). The Bárány inequality (H6) is proved in Section 6, and the remaining claims (H2), (H3) and (H4) follow from our discussion of box complexes in Section 5.

Remarks on gaps/tightness.

(H1) The gap in the first inequality may be arbitrarily large. For example, for graphs without a 4-cycle, which can have arbitrarily large chromatic number, all the topological lower bounds presented here are trivial. Indeed, for every G without a 4-cycle, there is a canonical $\mathbb{Z}_2$-equivariant $\mathbb{Z}_2$-map of $\operatorname{sd} \mathsf{B}(G)$ to a 1-dimensional subcomplex of $\mathsf{B}(G)$, given by $A' \uplus A'' \mapsto \emptyset \uplus \mathrm{CN}(A')$ for $|A'| \geq 2$, $A' \uplus A'' \mapsto \mathrm{CN}(A'') \uplus \emptyset$ for $|A''| \geq 2$, and $A' \uplus A'' \mapsto A' \uplus A''$ otherwise, and we get $\operatorname{ind} \mathsf{B}(G) \leq 1$.

(H3) The gap in the inequality (H3) can be at most 1; this can be derived from the inequality (M1) of Proposition 4 below, which yields $\operatorname{ind} \mathsf{B}_0(G) \geq \operatorname{ind} \mathsf{B}(G)$.

(H6) The Bárány bound can be strictly larger than the Dol'nikov–Kříž bound, without any bound on the gap, as will be discussed at the end of Section 6 for the example of the Schrijver graphs. Thus, in particular, the gap in (H3) can be arbitrarily large.

The Bárány bound depends on the choice of the polytope P. At present we do not know whether P can always be chosen of dimension at most $\dim \mathsf{K}^{*2}_{\Delta} + 1$, that is, whether the Bárány bound can always be made at least as strong as the Dol'nikov–Kříž bound. On the other hand, we do not have an example where the Bárány bound is necessarily smaller than the Sarkaria bound.

*We have labelled the equations and inequalities in the following chain of by (H1)–(H7); we will refer to these labels below when we prove the relations, one by one. Note that this is a chain of inequalities and equations, except at the end, where we do not imply a relation between the Bárány bound and the Dol'nikov–Kříž bound.

Remarks on size/computability. Although the Doľnikov-Kříž "colorability defect" bound is attractive since it is *combinatorial*, in general the Lovász bound may be much tighter. However, the number of vertices of $\mathsf{L}(G)$ may be exponential in $n = |V(G)|$, and similarly for some of the other box complexes. On the other hand, $\mathsf{B}(G)$ has only $2n$ vertices. The number of simplices can still be exponential, but if, for example, the maximum degree of G is bounded by a constant, then there are at most polynomially many simplices. Perhaps a computation of the $\mathbb{Z}_2$-acyclicity of $\mathsf{B}(G)$, which provides a lower bound for $\operatorname{ind}\mathsf{B}(G)$ (and thus for $\chi(G)$), might be feasible in some cases.

5 Box complexes and neighborhood complexes

In the following definition, we collect six (natural) variants of box complexes, four defined for a graph and two for a Kneser representation of it. For completeness, we also include the box complexes $\mathsf{B}(G)$ and $\mathsf{B}_0(G)$ that were already defined above.

Definition 2 (Box complexes). Let $G = (V, E) = \mathrm{KG}(\mathcal{F})$ be a finite (Kneser) graph with no isolated vertices, and suppose that the ground set of $\mathcal{F}$ is $[n]$. The first two complexes are on the vertex set $V\times[2]$; they were already defined in Section 3.

1. The box complex $\mathsf{B}(G)$ is

$$\begin{aligned}\mathsf{B}(G) \;&:=\; \big\{A' \uplus A'' : A', A'' \subseteq V,\ A' \cap A'' = \emptyset,\\ &\qquad\qquad G[A', A''] \text{ is complete},\ \mathrm{CN}(A'), \mathrm{CN}(A'') \neq \emptyset\big\}.\end{aligned}$$

   Equivalently, but more concisely, we can also write $\mathsf{B}(G) = \big\{A' \uplus A'' : A' \subseteq \mathrm{CN}(A'') \neq \emptyset,$ $A'' \subseteq \mathrm{CN}(A') \neq \emptyset\big\}$.

2. A simpler definition, but a larger complex, is obtained as

$$\mathsf{B}_0(G) \;:=\; \big\{A' \uplus A'' : A', A'' \subseteq V,\ A' \cap A'' = \emptyset,\ G[A', A''] \text{ is complete}\big\}.$$

   This is almost as for $\mathsf{B}(G)$, but here if one shore is empty, the other can be anything.

3. The following definition of a box complex, from Kříž [13, p. 568], takes into account only the complete bipartite graphs with *both* shores A' and A'' nonempty:

$$\begin{aligned}\mathsf{B}_{\text{chain}}(G) \;&:=\; \Delta\big\{A' \uplus A'' : \emptyset \neq A', A'' \subset V,\\ &\qquad\qquad A' \cap A'' = \emptyset,\ G[A', A''] \text{ is complete}\big\}.\end{aligned}$$

   Here the *vertices* are the vertex sets of complete bipartite subgraphs of G, and the simplices are chains of such sets under inclusion.

4. The vertices of the next box complex, from Alon, Frankl & Lovász [1, p. 361], are directed edges of G; that is, ordered pairs (u, v) with $\{u, v\} \in E$. We let

$$\mathsf{B}_{\text{edge}}(G) \;:=\; \big\{\vec{F} \subseteq A' \times A'' : \emptyset \neq A', A'' \subset V,\ A' \cap A'' = \emptyset,\ G[A', A''] \text{ is complete}\big\}.$$

   That is, simplices are subsets of edge sets of complete bipartite subgraphs of G, where the edges are oriented from the first shore to the second shore.

5. The simplicial complex $\Delta\big((2^{[n]})^{*2}_{\Delta}\backslash \mathsf{K}^{*2}_{\Delta}\big)$ appearing in Theorem 1, used as an intermediate step by Sarkaria in the derivation of his lower bounds, can be more explicitly written as

$$\begin{aligned}\mathsf{B}^{\mathrm{KG}}_{\mathrm{Sark}}(\mathcal{F}) \quad := \quad & \Delta\big\{B' \uplus B'' : B', B'' \subseteq [n],\ B' \cap B'' = \emptyset, \\ & \qquad\qquad \text{at least one of } B', B'' \text{ contains a set of } \mathcal{F}\big\}.\end{aligned}$$

The vertex set are pairs of disjoint subsets of the ground set of $\mathcal{F}$ that support a complete bipartite subgraph of the Kneser graph, with at least one shore nonempty.

6. Finally, another Kneser box complex, as in Kříž [13, p. 574], is

$$\begin{aligned}\mathsf{B}^{\mathrm{KG}}_{\mathrm{chain}}(\mathcal{F}) \quad := \quad & \Delta\big\{B' \uplus B'' : B', B'' \subseteq [n],\ B' \cap B'' = \emptyset, \\ & \qquad\qquad \text{both } B', B'' \text{ contain a set of } \mathcal{F}\big\}.\end{aligned}$$

On each of these types of box complexes, we have the natural $\mathbb{Z}_2$-action that interchanges the shores of the bipartite subgraph.

As we will show, all these box complexes fall into two groups, and those in each group have the same $\mathbb{Z}_2$-index. Moreover, the Lovász complex $\mathsf{L}(G)$ can also be included in one of the groups.

Theorem 3. *The following holds for the $\mathbb{Z}_2$-indices of the various box complexes:*

$$\begin{aligned}\operatorname{ind}\mathsf{B}_{\mathrm{chain}}(G) \;&=\; \operatorname{ind}\mathsf{B}^{\mathrm{KG}}_{\mathrm{chain}}(\mathcal{F}) \;=\; \operatorname{ind}\mathsf{B}_{\mathrm{edge}}(G) \;=\; \operatorname{ind}\mathsf{B}(G) \;=\; \operatorname{ind}\mathsf{L}(G) \\ &\le\; \operatorname{ind}\mathsf{B}_0(G) \;=\; \operatorname{ind}\mathsf{B}^{\mathrm{KG}}_{\mathrm{Sark}}(\mathcal{F}) \;\le\; \operatorname{ind}\mathsf{B}_{\mathrm{chain}}(G)+1.\end{aligned}$$

For a proof of this theorem, the following proposition provides explicit simplicial $\mathbb{Z}_2$-maps among the various box complexes. Here $\operatorname{susp}\mathsf{K}$ denotes the *suspension* of a simplicial complex K (a "double cone" over K): $\operatorname{susp}\mathsf{K} := \mathsf{K} \cup \{S \cup \{s\} : S \in \mathsf{K}\} \cup \{S \cup \{n\} : S \in \mathsf{K}\}$, where s and n are two new vertices not belonging to $V(\mathsf{K})$. The last inequality of the theorem follows from the map (M4) in the proposition, together with $\operatorname{ind}\operatorname{susp}\mathsf{K} \le \operatorname{ind}\mathsf{K}+1$ and the fact that $\operatorname{susp} S^n$ is an S^{n+1}.

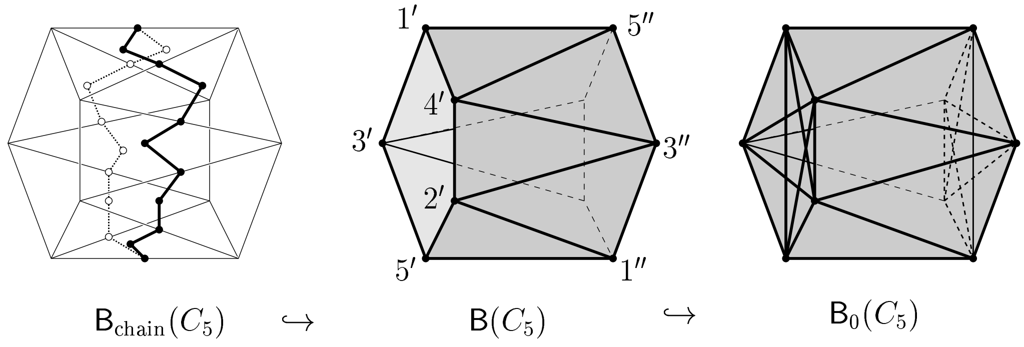
Proposition 4 ($\mathbb{Z}_2$-maps). *For every finite graph $G = \mathrm{KG}(\mathcal{F})$ without isolated vertices, there are canonical simplicial $\mathbb{Z}_2$-maps*

$$\mathsf{B}(G) \longrightarrow \mathsf{B}_0(G), \tag{M1}$$
$$\operatorname{sd}\mathsf{B}_{\mathrm{edge}}(G) \longleftrightarrow \mathsf{B}_{\mathrm{chain}}(G), \tag{M2}$$
$$\mathsf{B}_{\mathrm{chain}}(G) \longrightarrow \operatorname{sd}\mathsf{B}(G), \tag{M3}$$
$$\operatorname{sd}\mathsf{B}_0(G) \longrightarrow \operatorname{susp}\mathsf{B}_{\mathrm{chain}}(G), \tag{M4}$$
$$\mathsf{B}_{\mathrm{chain}}(G) \longleftrightarrow \mathsf{B}^{\mathrm{KG}}_{\mathrm{chain}}(\mathcal{F}), \tag{M5}$$
$$\operatorname{sd}\mathsf{B}_0(G) \longleftrightarrow \mathsf{B}^{\mathrm{KG}}_{\mathrm{Sark}}(\mathcal{F}), \tag{M6}$$
$$\operatorname{sd}\operatorname{sd}\mathsf{B}(G) \longrightarrow \mathsf{B}_{\mathrm{chain}}(G), \tag{M7}$$
$$\operatorname{sd}\mathsf{L}(G) \longrightarrow \mathsf{B}_{\mathrm{chain}}(G), \tag{M8}$$
$$\operatorname{sd}\mathsf{B}_{\mathrm{chain}}(G) \longrightarrow \operatorname{sd}\mathsf{L}(G). \tag{M9}$$

Answering a question by the first author, Lovász (personal communication, February 2000) proved that $\mathsf{L}(G)$ is homotopy equivalent to $\mathsf{B}_{\text{chain}}(G)$ (using the nerve theorem); our construction of the $\mathbb{Z}_2$-map in (M9) is inspired by his proof.

After a preliminary version of this paper was written, Csorba [7] proved that $\mathsf{B}_0(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $\operatorname{susp}(\mathsf{B}(G))$ for every G, thereby considerably strengthening our statements above concerning these box complexes.

The following figure sketches the three main box complexes for $G = C_5$ and suggests the maps between them. Here $\mathsf{B}(C_5)$ is homeomorphic to $S^1 \times I$; it is a subcomplex of $\mathsf{B}_0(C_5)$, which additionally contains two simplices on 5 vertices. The complex $\mathsf{B}_{\text{chain}}(C_5)$, an S^1 on 10 vertices, embeds into the barycentric subdivision of $\mathsf{B}(C_5)$:



Proof.

(M1) This map is simply the identity on the vertex set: an inclusion map.

(M2) The map $\operatorname{sd}\mathsf{B}_{\text{edge}}(G) \to \mathsf{B}_{\text{chain}}(G)$ is defined on the vertices of $\operatorname{sd}\mathsf{B}_{\text{edge}}(G)$ by setting $\vec{F} \mapsto h(\vec{F}) \uplus t(\vec{F})$, where $h(\vec{F})$ collects the tails, and $t(\vec{F})$ collects the heads, of the directed edges in $\vec{F}$.

The map $\mathsf{B}_{\text{chain}}(G) \to \operatorname{sd}\mathsf{B}_{\text{edge}}(G)$ in the other direction is obtained by sending $A' \uplus A''$ to the (directed) edge set of the complete bipartite graph with shores A' and A''.

(M3) The map $\mathsf{B}_{\text{chain}}(G) \to \operatorname{sd}\mathsf{B}(G)$ is simply an inclusion, as each vertex of $\mathsf{B}_{\text{chain}}(G)$ is also a simplex of $\mathsf{B}(G)$.

(M4) The map $\operatorname{sd}\mathsf{B}_0(G) \to \operatorname{susp}\mathsf{B}_{\text{chain}}(G)$ is obtained by mapping the "improper" complete bipartite subgraphs (with one shore empty) to the suspension points.

(M5) A canonical $\mathbb{Z}_2$-map $\mathsf{B}^{\text{KG}}_{\text{chain}}(\mathcal{F}) \to \mathsf{B}_{\text{chain}}(G)$ is defined on the vertices by

$$B' \uplus B'' \longmapsto \{F' \in \mathcal{F} : F' \subseteq B'\} \uplus \{F'' \in \mathcal{F} : F'' \subseteq B''\}.$$

A canonical $\mathbb{Z}_2$-map $\mathsf{B}_{\text{chain}}(G) \to \mathsf{B}^{\text{KG}}_{\text{chain}}(\mathcal{F})$ is obtained by mapping the vertices:

$$A' \uplus A'' \longmapsto \left(\bigcup A'\right) \uplus \left(\bigcup A''\right).$$

(M6) The same formulas define $\mathbb{Z}_2$-maps $\mathsf{B}^{\text{KG}}_{\text{Sark}}(\mathcal{F}) \longleftrightarrow \operatorname{sd}\mathsf{B}_0(\mathrm{KG}(\mathcal{F}))$.

(M7) The map $\operatorname{sd}\operatorname{sd}\mathsf{B}(G) \to \mathsf{B}_{\text{chain}}(G)$ is defined on the second barycentric subdivision of $\mathsf{B}(G)$, and so its *vertices* are chains of the form

$$\mathcal{A} = \big(A'_0 \uplus A''_0 \subset \cdots \subset \cdots \subset A'_k \uplus A''_k\big).$$

We let $\mu'(\mathcal{A})$ be the smallest nonempty set in the chain of sets $\mathcal{A}' := (A'_0 \subset \cdots \subset A'_k \subseteq \mathrm{CN}(A''_k) \subseteq \cdots \subseteq \mathrm{CN}(A''_0))$, and similarly for $\mu''(\mathcal{A})$ (here we use the condition $\mathrm{CN}(A'), \mathrm{CN}(A'') \neq \emptyset$ from the definition of $\mathsf{B}(G)$). We let the image of $\mathcal{A}$ be $\mu'(\mathcal{A}) \uplus \mu''(\mathcal{A})$. This is a vertex of $\mathsf{B}_{\text{chain}}(G)$: Since the first barycentric subdivision does not contain $\emptyset$ as a vertex, at least one of A'_0, A''_0, say A'_0, is nonempty. Then we have $\mu'(\mathcal{A}) = A'_0$, while $\mu''(\mathcal{A})$ is contained in $\mathrm{CN}(A''_0)$. If we extend the chain $\mathcal{A}$, then this also leads to an extension of the chains $\mathcal{A}'$ and $\mathcal{A}''$, so the sets $\mu'(\mathcal{A})$ and $\mu''(\mathcal{A})$ can only get smaller. Therefore, our map is simplicial, and it is clearly a $\mathbb{Z}_2$-map.

(M8) We recall that the vertices of $\mathsf{L}(G)$ are the nonempty subsets $A \subset V$ that are *closed* in the sense that $A = \mathrm{CN}(\mathrm{CN}(A))$, or equivalently, $A = \mathrm{CN}(B)$ for some nonempty subset $B \subset V$.

First we define a $\mathbb{Z}_2$-map $f\colon \operatorname{sd}\mathsf{L}(G) \to \mathsf{B}_{\text{chain}}(G)$. A vertex of $\operatorname{sd}\mathsf{L}(G)$ is a chain $\mathcal{A} = (A_0 \subset A_1 \subset \cdots \subset A_k)$ of nonempty closed sets. We set $f(\mathcal{A}) := A_0 \uplus \mathrm{CN}(A_k)$. Since $\mathrm{CN}(A_k) \subseteq \mathrm{CN}(A_0)$, the image is indeed a vertex of $\mathsf{B}_{\text{chain}}(G)$. If a chain $\mathcal{A}'$ extends $\mathcal{A}$, its first set can only be smaller than the first set of $\mathcal{A}$, and the last set can only be larger than the last set of $\mathcal{A}$. Therefore, $f(\mathcal{A}') \subseteq f(\mathcal{A})$, and it follows that f is simplicial. Finally, the image of $\mathcal{A}$ under the $\mathbb{Z}_2$-action on $\operatorname{sd}\mathsf{L}(G)$ is the chain $\mathcal{B} = (\mathrm{CN}(A_k) \subset \mathrm{CN}(A_{k-1}) \subset \cdots \subset \mathrm{CN}(A_0))$. We have $f(\mathcal{A}) = A_0 \uplus \mathrm{CN}(A_k)$ and $f(\mathcal{B}) = \mathrm{CN}(A_k) \uplus \mathrm{CN}^2(A_0) = \mathrm{CN}(A_k) \uplus A_0$ (as A_0 is closed), and so f is a $\mathbb{Z}_2$-map.

(M9) Finally, we provide a $\mathbb{Z}_2$-map $\operatorname{sd}\mathsf{B}_{\text{chain}}(G) \to \operatorname{sd}\mathsf{L}(G)$. A vertex in $\operatorname{sd}\mathsf{B}_{\text{chain}}(G)$ is a chain $\mathcal{A} = \big(A'_0 \uplus A''_0 \subset \cdots \subset A'_k \uplus A''_k\big)$. All the sets $\mathrm{CN}^2(A'_0), \ldots, \mathrm{CN}^2(A'_k)$, $\mathrm{CN}(A''_0), \ldots, \mathrm{CN}(A''_k)$ are closed and nonempty, and the following inclusion are easily verified: $\mathrm{CN}^2(A'_0) \subseteq \cdots \subseteq \mathrm{CN}^2(A'_k) \subseteq \mathrm{CN}(A''_k) \subseteq \cdots \subseteq \mathrm{CN}(A''_0)$. So by omitting repeated sets from this chain, we obtain a vertex of $\operatorname{sd}\mathsf{L}(G)$. The chain $\mathcal{A}$ is mapped to this vertex. If we extend $\mathcal{A}$, the image stays the same or is extended as well, so the map is simplicial. Finally, it is a $\mathbb{Z}_2$-map; here we use that $\mathrm{CN}^3 = \mathrm{CN}$. $\square$

The above definitions by far do not exhaust the list of possible (and possibly interesting) box complex variants. For example, we could also define the "Kneser counterparts" of $\mathsf{B}(G)$ and $\mathsf{B}_0(G)$, namely $\mathsf{B}^{\mathrm{KG}}(\mathcal{F}) := \big\{S \subseteq B' \uplus B'' : B', B'' \subseteq [n], B' \cap B'' = \emptyset$, both B' and B'' contain a set of $\mathcal{F}\big\}$ and $\mathsf{B}_0^{\mathrm{KG}}(\mathcal{F}) := \big\{S \subseteq B' \uplus B'' : B', B'' \subseteq [n], B' \cap B'' = \emptyset$, at least one of B', B'' contains a set of $\mathcal{F}\big\}$. By similar arguments as in the proof of Proposition 4, it can be shown that $\operatorname{ind}\mathsf{B}_0(\mathrm{KG}(\mathcal{F})) = \operatorname{ind}\mathsf{B}_0^{\mathrm{KG}}(\mathcal{F})$ and $\operatorname{ind}\mathsf{B}(\mathrm{KG}(\mathcal{F})) = \operatorname{ind}\mathsf{B}^{\mathrm{KG}}(\mathcal{F})$, but we prefer to omit this part.

In some of the cases in Proposition 4, the maps even provide $\mathbb{Z}_2$-homotopy equivalences between the respective complexes. Since the proofs are not very interesting and, at present, the $\mathbb{Z}_2$-homotopy types do not seem to bring anything new concerning the lower bounds for the chromatic number, we have decided not to discuss this in the present paper.

The box complexes of complete graphs. In order to use a box complex for bounding the chromatic number of a graph, we need to know the $\mathbb{Z}_2$-index of the box complex of K_m. In view of the equalities of indices established above for the various box complexes, we mention only two of the box complexes here (but the others can also be analyzed directly without difficulty).

Proposition 5. *The polyhedron of the simplicial complex* $\mathsf{B}_0(K_m)$ *is (homeomorphic to) an* S^{m-1}, *and thus* $\operatorname{ind}\mathsf{B}_0(K_m) = m-1$.

The polyhedron of the simplicial complex $\mathsf{B}(K_m)$ *is homeomorphic to* $S^{m-2}\times[0,1]$, *and thus* $\operatorname{ind}\mathsf{B}(K_m) = m-2$.

Proof. $\mathsf{B}_0(K_m)$ is isomorphic to the boundary complex of an m-dimensional cross polytope ("generalized octahedron," or unit ball of the ℓ_1-norm), with its canonical antipodal $\mathbb{Z}_2$-action.

The complex $\mathsf{B}(K_m)$ is isomorphic to the boundary complex of an m-dimensional cross polytope with two opposite facets removed, and with its canonical antipodal $\mathbb{Z}_2$-action. $\square$

By now, all parts of Theorem 1 have been proved, except for the claims involving the Bárány bound.

6 The Bárány bound

Proof of the inequality (H6). We need to prove that if the simplicial complex K is a part of the boundary of an k-dimensional convex polytope P, then $\operatorname{ind}\mathsf{K}^{*2}_{\Delta} \le k-1$. Briefly speaking, the reason is that the deleted join of S^{k-1} contains $\|\mathsf{K}^{*2}_{\Delta}\|$ and is $\mathbb{Z}_2$-homotopy equivalent to S^{k-1}. A more detailed argument follows.

Let us recall that if X and Y are topological spaces, the join $X * Y$ is the quotient of the product space $X\times Y\times[0,1]$ by the equivalence $\approx$, where $(x,y,0) \approx (x',y,0)$ and $(x,y,1) \approx (x,y',1)$ for all $x,x' \in X$ and $y,y' \in Y$. If K is a simplicial complex and $\mathsf{K}^{*2} := \{S \uplus S' : S,S' \in \mathsf{K}\}$, then there is a (canonical) homeomorphism $\|\mathsf{K}\|^{*2} \cong \|\mathsf{K}^{*2}\|$.

Here we need the space

$$Z \ := \ \left(S^{k-1} * S^{k-1}\right) \setminus \{(x,x,\tfrac{1}{2}) : x \in S^{k-1}\};$$

this is a kind of deleted join of S^{k-1} (considered as a topological space, not a simplicial complex). The space Z is equipped with the $\mathbb{Z}_2$ action given by $(x,y,t) \mapsto (y,x,1-t)$.

Using the canonical homeomorphism $\|\mathsf{K}^{*2}\| \to \|\mathsf{K}\|^{*2}$ and a homeomorphism of the boundary of P with S^{k-1}, we obtain a $\mathbb{Z}_2$-map $h\colon \|\mathsf{K}^{*2}\| \to S^{k-1} * S^{k-1}$. Moreover, if we restrict the left-hand side to $\|\mathsf{K}^{*2}_{\Delta}\|$, then the image contains no point of the form $(x,x,\frac{1}{2})$, and so h is a $\mathbb{Z}_2$-map $\|\mathsf{K}^{*2}_{\Delta}\| \to Z$. So it suffices to show that $\operatorname{ind} Z \le k-1$. Let S^{k-1} be represented as the unit sphere in $\mathbb{R}^k$. We define a $\mathbb{Z}_2$-map $Z \to S^{k-1}$ by

$$(x,y,t) \mapsto \frac{tx - (1-t)y}{\|tx - (1-t)y\|}.$$

If $tx - (1-t)y = 0$ for unit vectors x,y, then $t = \frac{1}{2}$ and $x = y$. Thus, the map is well defined, and one can also check that it is continuous. Clearly, it commutes with the respective $\mathbb{Z}_2$-actions. This concludes the proof. $\square$

An extension of Bárány's proof. Here we directly prove the inequality $\chi(\mathrm{KG}(\mathcal{F})) \ge d$ whenever $\mathsf{K}(\mathcal{F})$ is a part of the boundary of an $(n-d)$-dimensional simplicial convex polytope. This explains the relation of this bound to Bárány's [4] original proof of the Kneser conjecture.

Let $V = (v_1, v_2, \ldots, v_n) \subset \mathbb{R}^{n-d}$ be the vertex set of P, and let $V^* = (v_1^*, v_2^*, \ldots, v_n^*) \subset \mathbb{R}^{d-1}$ be a Gale diagram of V (see, e. g., [24, Lect. 6] for an introduction to Gale diagrams). Without loss of generality, we may assume that all points in V^* different from 0 lie in S^{d-2}. We recall that the points of V, and thus also the points of V^*, are in one-to-one correspondence with the elements of the ground set of $\mathcal{F}$.

We want to show that *every open hemisphere in this S^{d-2} contains a set corresponding to a set of $\mathcal{F}$*. Once this is done, we can proceed exactly as in Bárány's proof. Namely, supposing that the sets of $\mathcal{F}$ have been colored by at most $d-1$ colors, we define the set $A_i \subseteq S^{d-2}$ as the set of all $x \in S^{d-2}$ such that the open hemisphere centered at x contains at least one set of $\mathcal{F}$ colored by i, $i \in [d-1]$. Each A_i is open, and by the above claim, the A_i together cover all of S^{d-2}. By a suitable version of the Borsuk–Ulam theorem (called the Lyusternik–Shnirel'man theorem), there exist $x \in S^{d-2}$ and $i \in [d-1]$ such that $x, -x \in A_i$. This means that two opposite open hemispheres both contain a set of color i, and so the coloring is not a proper coloring of the Kneser graph.

To prove the claim, consider an open hemisphere H, and let $S^* := H \cap V^*$. Since H is defined by an open halfspace, there is a linear functional on $\mathbb{R}^{d-1}$ that is positive on S^* and nonpositive on $V^* \setminus S^*$. Let $S \subset V$ be the set corresponding to S^*. Properties of the Gale diagram imply that there is an affine dependence of the points of V in which the points of S have positive coefficients and the other points have nonpositive coefficients. This means that $\operatorname{conv}(S) \cap \operatorname{conv}(V \setminus S) \neq \emptyset$. So S is not the vertex set of a face of P, and thus $S \notin \mathsf{K}(\mathcal{F})$. This means that S contains a set of $\mathcal{F}$. The proof is finished. $\square$

On Schrijver graphs and the Dol'nikov–Kříž bound. Let $0 < 2k < n$, and let $\binom{[n]}{k}_{\text{stab}}$ denote the system of all sets $F \subseteq [n]$ such that if $i \in F$ then $i+1 \notin F$, and if $n \in F$ then $1 \notin F$. (So the sets of $\binom{[n]}{k}_{\text{stab}}$ can be identified with the independent sets in the cycle of length n, with the numbering of vertices from 1 to n along the cycle.) Schrijver [21] proved that the graph $\mathrm{SG}_{n,k} := \mathrm{KG}\big(\binom{[n]}{k}_{\text{stab}}\big)$ is a vertex-critical subgraph of the Kneser graph $\mathrm{KG}\big(\binom{[n]}{k}\big)$. That is, $\chi(\mathrm{SG}_{n,k}) = n - 2k + 2$, and every proper induced subgraph of $\mathrm{SG}_{n,k}$ has a smaller chromatic number. The criticality follows by a clever coloring construction and we will not consider it here; we look at a proof of the lower bound.

It turns out that the Bárány bound applies very neatly here. Let $P := C_{2k-2}(n)$ be a cyclic polytope of dimension $2k-2$ on n vertices. With the usual numbering of the vertices, Gale's evenness criterion (see [24, Thm. 0.7]) shows that the $S \subseteq [n]$ that contain no set of $\binom{[n]}{k}_{\text{stab}}$ are exactly the proper faces of P. Therefore, $\mathsf{K}\big(\binom{[n]}{k}_{\text{stab}}\big) = \partial P$, and the Bárány bound immediately yields $\chi(\mathrm{SG}_{n,k}) \geq n - 2k + 2$.

The Schrijver graphs also provide examples where the Dol'nikov–Kříž bound is considerably weaker than the Bárány bound. Indeed, it is easy to check that $\mathrm{cd}_2\big(\binom{[n]}{k}_{\text{stab}}\big) = n-4k+4$.

7 Concluding remarks

1. Many of the above considerations can easily be extended to Kneser hypergraphs (where vertices are again the sets of $\mathcal{F}$, and edges are r-tuples of pairwise disjoint sets, for some given r), and even to the s-disjoint Kneser hypergraph version of Sarkaria, as in Ziegler [25]. A detailed exploration of this is a subject for further research.

   Indeed, the p-partite versions of some of our box complexes as presented in Section 5 appear in the published literature that concerns Kneser hypergraphs: The p-partite version of $\mathsf{B}_{\text{edge}}(G)$ is used in Alon, Frankl & Lovász [1, p. 361]. The p-partite version of $\mathsf{B}_{\text{chain}}(G)$ appears in Kříž [13, p. 568], while in Kříž [13, p. 574] we find the p-partite version of $\mathsf{B}^{\text{KG}}_{\text{chain}}(G)$. The box complexes appear in Kříž' work as "resolution complexes" for equivariant cohomology.

2. In a discussion with Jarik Nešetřil, we noted the following interpretation of $\mathsf{B}_0(G)$ (and $\mathsf{B}(G)$): the simplices of $\mathsf{B}_0(G)$ are vertex sets of complete bipartite subgraphs in $G \times K_2$, where the "categorical product" of graphs $G \times H$ has vertex set $V(G) \times V(H)$ and edges $\{(u, v), (u', v')\}$ such that $\{u, u'\} \in E(G)$ and $\{v, v'\} \in E(H)$. The $\mathbb{Z}_2$-action is then induced by the exchange of vertices of K_2. A homomorphism $G \to G'$ induces a homomorphism $G \times K_2 \to G' \times K_2$ (so the $\mathsf{B}(\cdot)$ functor "factors" in this way). This can be generalized, for example, to a product $G \times C_p$, where C_p is the p-cycle with the natural $\mathbb{Z}_p$-action (cyclic shift).

3. In a similar spirit but earlier, Lovász and others have considered a setting of "Hom-complexes": For this let H and G be finite graphs, and let Δ_G be a simplex with vertex set $V(G)$. The Cartesian power $(\Delta_G)^{|V(H)|}$ is a polyhedral complex, whose vertices can be identified with mappings $V(H) \to V(G)$. Then $\text{Hom}(H, G)$ is the subcomplex induced by the vertices that are homomorphisms (thus, the faces are all polyhedra $F \in (\Delta_G)^{|V(H)|}$ such that each vertex of F is a homomorphism). This rather general setting covers Lovász' connectivity lower bound, as well as some of the recent work of Brightwell and Winkler [6]. Lovász conjectured that if $\text{Hom}(C_{2r+1}, G)$ is k-connected then $\chi(G) \geq k + 4$. Here $r \geq 1$ is an arbitrary integer and C_{2r+1} denotes the odd cycle of length $2r + 1$. This was recently proved by Babson and Kozlov [3] by advanced topological methods. Let us remark that while $\chi(G) \geq k + 3$ is easy to establish by methods discussed in the present paper, the improvement by 1 currently seems hard, and it represents new kind of topological obstruction to $(k + 4)$-colorability, which apparently is not captured by any of the lower bounds discussed above.

4. The inequality $\operatorname{ind}\operatorname{susp}\mathsf{K} \leq \operatorname{ind}\mathsf{K}$ may be strict for finite simplicial complexes K: For example, one may obtain a cell complex model by taking $h\colon S^3 \to S^2$ to be the Hopf map, and attaching two 4-cells to S^2 via $2h$ resp. $-2h$, where multiples of maps are taken according to addition in $\pi_3(S^2) \cong \mathbb{Z}$. (This particular example was suggested by Péter Csorba, based on an earlier construction by Csorba, Živaljević, and the first author for a different purpose; another approach was proposed by Wojchiech Chachólski.)

   However, it is not clear whether $\operatorname{ind}\operatorname{susp}\mathsf{K} = \operatorname{ind}\mathsf{K}$ occurs in the rather special setting of (H3), where K is a box complex. An interesting step in this direction was recently taken by Csorba (private communication), who proved that given any simplicial $\mathbb{Z}_2$-complex K, there is a graph G such that $\mathsf{N}(G)$ is homotopy equivalent to K. At present it is not clear whether this result can be extended to a $\mathbb{Z}_2$-homotopy equivalence of the box complex of a suitable graph with a given K; if yes, this would provide many pathological examples and, in particular, it would answer the above question.

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