What Was It About? An Informal Summary

Chapter 1

- Linear and affine notions (dependence, hull, subspace, mapping); hyperplane, k-flat.
- General position: Degenerate configurations have measure zero in the space of all configurations, provided that degeneracy can be described by countably many polynomial equations.
- Convex set, hull, combination.
- Separation theorem: Disjoint convex sets can be separated by a hyperplane; strictly so if one of them is compact and the other closed.
- Theorems involving the dimension: Helly (if $\mathcal{F}$ is a finite family of convex sets with empty intersection, then there is a subfamily of at most $d{+}1$ sets with empty intersection), Radon ($d{+}2$ points can be partitioned into two subsets with intersecting convex hulls), Carathéodory (if $x \in \operatorname{conv}(X)$, then $x \in \operatorname{conv}(Y)$ for some at most $(d{+}1)$-point $Y \subseteq X$).
- Centerpoint of X: Every half-space containing it contains at least $\frac{1}{d+1}$ of X. It always exists by Helly. Ham-sandwich: Any d mass distributions in $\mathbf{R}^d$ can be simultaneously bisected by a hyperplane.

Chapter 2

- Minkowski's theorem: A 0-symmetric convex body of volume larger than 2^d contains a nonzero integer point.
- General lattice: a discrete subgroup of $(\mathbf{R}^d, +)$. It can be written as the set of all integer linear combinations of at most d linearly independent vectors (basis). Determinant = volume of the parallelotope spanned by a basis.
- Minkowski for general lattices: Map the lattice onto $\mathbf{Z}^d$ by a linear mapping.

Chapter 3

- Erdős–Szekeres theorem: Every sufficiently large set in the plane in general position contains k points in convex position. How large? Exponential in k.
- What about k-holes (vertex sets of empty convex k-gons)? For $k = 5$ yes (in sufficiently large sets), for $k \geq 7$ no (Horton sets), $k = 6$ is a challenging open problem.

Chapter 4

- Szemerédi–Trotter theorem: m distinct points and n distinct lines in the plane have at most $O(m^{2/3}n^{2/3} + m + n)$ incidences.
- This is tight in the worst case. Example for $m = n$: Use the $k \times 4k^2$ grid and lines $y = ax + b$ with $a = 0, 1, \ldots, 2k-1$ and $b = 0, 1, \ldots, 2k^2-1$.
- Crossing number theorem: A simple graph with n vertices and $m \geq 4n$ edges needs $\Omega(m^3/n^2)$ crossings. Proof: At least $m-3n$ crossings, since planar graphs have fewer than $3n$ edges, then random sampling.
- Forbidden bipartite subgraphs: A graph on n vertices without $K_{r,s}$ has $O(n^{2-1/r})$ edges.
- Cutting lemma: Given n lines and r, the plane can be subdivided into $O(r^2)$ generalized triangles such that the interior of each triangle is intersected by at most $\frac{n}{r}$ lines. Proof of a weaker version: Triangulate the arrangement of a random sample and show that triangles intersected by many lines won't survive. Application: geometric divide-and-conquer.
- For unit distances and distinct distances in the plane, bounds can be proved, but a final answer seems to be far away.

Chapter 5

- Geometric duality: Sends a point a to the hyperplane $\langle a, x \rangle = 1$ and vice versa; preserves incidences and sidedness.
- Convex polytope: the convex hull of a finite set and also the intersection of finitely many half-spaces.
- Face, vertex, edge, facet, ridge. A polytope is the convex hull of its vertices. A face of a face is a face. Face lattice. Duality turns it upside down. Simplex. Simple and simplicial polytopes.
- The convex hull of n points in $\mathbf{R}^d$ can have as many as $\Omega(n^{\lfloor d/2 \rfloor})$ facets; cyclic polytopes.
- This is as bad as it can get: Given the number of vertices, cyclic polytopes maximize the number of faces in each dimension (upper bound theorem).
- Gale transform: An n-point sequence in $\mathbf{R}^d$ (affinely spanning $\mathbf{R}^d$) is mapped to a sequence of n vectors in $\mathbf{R}^{n-d-1}$. Properties: a simple linear algebra. Faces of the convex hull go to subsets whose complement contains 0 in the convex hull.

- 3-dimensional polytopes are nice: Their graphs correspond to vertex 3-connected planar graphs (Steinitz theorem), and they can be realized with rational coordinates. From dimension 4 on, bad things can happen (irrational or doubly exponential coordinates may be required, recognition is difficult).
- Voronoi diagram. It is the projection of a convex polyhedron in dimension one higher (lifting using the paraboloid). Delaunay triangulation (defined using empty balls; dual to the Voronoi diagram).

Chapter 6

- Arrangement of hyperplanes (faces, vertices, edges, facets, cells). For d fixed, there are $O(n^d)$ faces.
- Clarkson's theorem on levels: At most $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ vertices are at level at most k. Proof: Express the expected number of level-0 vertices of a random sample in two ways!
- Zone theorem: The zone of a hyperplane has $O(n^{d-1})$ vertices. Proof: Delete a random hyperplane, and look at how many zone faces are sliced into two by adding it back.
- Proof of the cutting lemma by a finer sampling argument: Vertically decompose the arrangement of a sample taken with probability p, show that the number of trapezoids intersected by at least tnp lines decreases exponentially with t, take $\frac{1}{t}$-cuttings within the trapezoids.
- Canonical triangulation, cutting lemma in $\mathbf{R}^d$ ($O(r^d)$ simplices).
- Milnor–Thom theorem: The arrangement of the zero sets of n polynomials of degree at most D in d real variables has at most $O(Dn/d)^d$ faces.
- Most arrangements of pseudolines are nonstretchable (by Milnor–Thom). Similarly for many other combinatorial descriptions of geometric configurations; usually most of them cannot be realized.

Chapter 7

- Davenport–Schinzel sequences of order s (no $abab\ldots$ with $s+2$ letters); maximum length $\lambda_s(n)$. Correspond to lower envelopes of curves: The curves are graphs of functions defined everywhere, every two intersecting at most s times. Lower envelopes of segments yield DS sequences of order 3.
- $\lambda_3 = \Theta(n\alpha(n))$; $\lambda_s(n)$ is almost linear for every fixed s.
- The lower envelope of n algebraic surface patches in $\mathbf{R}^d$, as well as a single cell in their arrangement, have complexity $O(n^{d-1+\varepsilon})$. Charging schemes and more random sampling.

Chapter 8

- Fractional Helly theorem: If a family of n convex sets has $\alpha\binom{n}{d+1}$ intersecting $(d+1)$-tuples, then there is a point common to at least $\frac{\alpha}{d+1}n$ of the sets.
- Colored Carathéodory theorem: If each of $d+1$ sets contains 0 in the convex hull, then we can pick one point from each set so that the convex hull of the picked points contains 0.
- Tverberg's theorem: $(d+1)(r-1)+1$ points can be partitioned into r subsets with intersecting convex hulls (the number is the smallest conceivable one: $r-1$ simplices plus one extra point).
- Colored Tverberg theorem: Given points partitioned into $d+1$ color classes by t points each, we can choose r disjoint rainbow subsets with intersecting convex hulls, $t = t(d,r)$. Only topological proofs are known.

Chapter 9

- The dimension is considered fixed in this chapter. First selection lemma: Given n points, there exists a point contained in a fixed fraction of all simplices with vertices in the given points.
- Second selection lemma: If $\alpha\binom{n}{d+1}$ of the simplices are marked, we can find a point in many of the marked simplices (at least $\Omega(\alpha^{s_d}\binom{n}{d+1})$). Needs colored Tverberg and Erdős–Simonovits.
- Order type. Same-type lemma: Given n points in general position and k fixed, one can find k disjoint subsets of size $\Omega(n)$, all of whose transversals have the same order type.
- A hypergraph regularity lemma: For an $\varepsilon > 0$ and a k-partite hypergraph of density bounded below by a constant $\beta > 0$ and with color classes $X_1, \ldots, X_n$ of size n, we can choose subsets $Y_1 \subseteq X_1, \ldots, Y_k \subseteq X_k$, $|Y_1| = \cdots = |Y_k| \geq cn$, $c = (k, \beta, \varepsilon) > 0$, such that any $Z_1 \subseteq Y_1, \ldots, Z_k \subseteq Y_k$ with $|Z_i| \geq \varepsilon|Y_i|$ induce some edge.
- Positive-fraction selection lemma: Given n red, n white, and n blue points in the plane, we can choose $\frac{n}{12}$ points of each color so that all red–white–blue triangles have a common point; similarly in $\mathbf{R}^d$.

Chapter 10

- Set systems; transversal number τ, packing number ν. Fractional transversal and fractional packing; $\nu^* = \tau^*$ by LP duality.
- Epsilon net, shattered set, VC-dimension. Shatter function lemma: A set system on n points with VC-dimension d has at most $\sum_{k=0}^{d}\binom{n}{k}$ sets.
- Epsilon net theorem: A random sample of $C\frac{d}{\varepsilon}\log\frac{1}{\varepsilon}$ points in a set system of VC-dimension d is an ε-net with high probability. In particular, ε-nets exist of size depending only on d and ε.
- Corollary: $\tau = O(\tau^* \log \tau^*)$ for bounded VC-dimension.

- Half-spaces in $\mathbf{R}^d$ have VC-dimension $d+1$. Lifting (Veronese map) and the shatter function lemma show that systems of sets in $\mathbf{R}^d$ definable by Boolean combinations of a bounded number of bounded-degree polynomial inequalities have bounded VC-dimension.
- Weak epsilon nets for convex sets: Convex sets have infinite VC-dimension, but given a finite set X and $\varepsilon > 0$, we can choose a weak ε-net of size at most $f(d, \varepsilon)$, that is, a set (generally not a subset of X) that intersects every convex C with $|C \cap X| \geq \varepsilon|X|$.
- Consequently, τ is bounded by a function of τ^* for any finite system of convex sets in $\mathbf{R}^d$.
- Alon–Kleitman (p, q)-theorem: Let $\mathcal{F}$ be a system of convex sets such that among every p sets, some q intersect ($p \geq q \geq d+1$). Then $\tau(\mathcal{F})$ is bounded by a function of d, p, q. Proof: First bound ν^* using fractional Helly; then τ is bounded in terms of $\tau^* = \nu^*$ as above.
- A similar (p, q)-theorem for hyperplane transversals of convex sets (even though no Helly theorem!).

Chapter 11

- k-sets, k-facets (only for sets in general position!), halving facets. Dual: cells of level k, vertices of level k. The k-set problem is still unsolved. Straightforward bounds from Clarkson's theorem on levels.
- Bounds for halving facets yield bounds for k-facets sensitive to k.
- A recursive planar construction with a superlinear number of halving edges.
- Lovász lemma: No line intersects more than $O(n^{d-1})$ halving facets. Proof: When a moving line crosses the convex hull of $d-1$ points of X, the number of halving facets intersected changes by 1 (halving-facet interleaving lemma).
- Implies an upper bound of $O(n^{d-\delta(d)})$ for halving facets by the second selection lemma.
- In the plane a continuous motion argument proves that the crossing number of the halving-edge graph is $O(n^2)$, and consequently, it has $O(n^{4/3})$ edges by the crossing number theorem. This is the best we can do in the plane, although $O(n^{1+\varepsilon})$ for every fixed $\varepsilon > 0$ is suspected.

Chapter 12

- Perfect graph ($\chi = \omega$ hereditarily). weak perfect graph conjecture (now theorem): A graph is perfect iff its complement is.
- Proof via the polytope $\{x \in \mathbf{R}^V : x \geq 0, x(K) \leq 1 \text{ for every clique } K\}$.
- Brunn's slice volume inequality: For a compact convex $C \subset \mathbf{R}^{n+1}$, $\mathrm{vol}_n(\{x \in C : x_1 = t\})^{1/n}$ is a concave function of t (as long as the slices do not miss the body).

- Brunn–Minkowski inequality: $\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \le \mathrm{vol}(A+B)^{1/n}$ for nonempty compact $A, B \subset \mathbf{R}^n$.
- A partially ordered set with N linear extensions can be sorted by $O(\log N)$ comparisons. There always exists a comparison that reduces the number of linear extensions by a fixed fraction: Compare elements whose average heights differ by less than 1.
- Order polytope: $0 \le x \le 1$, $x_a \le x_b$ whenever $a \preceq b$. Linear extensions correspond to congruent simplices and good comparison to dividing the volume evenly by a hyperplane $x_a = x_b$. The best ratio is not known (conjectured to be $\frac{1}{3} : \frac{2}{3}$).

Chapter 13

- Volumes and other things in high dimensions behave differently from what we know in $\mathbf{R}^2$ and $\mathbf{R}^3$. For example, the ball inscribed in the unit cube has a tiny volume.
- An η-net is an inclusion-maximal η-separated set. It is mainly useful because it is η-dense. In S^{n-1}, a simple volume argument yields η-nets of size at most $(4/\eta)^n$.
- An N-vertex convex polytope inscribed in the unit ball B^n occupies at most $O(\ln(\frac{N}{n}+1)/n)^{n/2}$ of the volume of B^n. Thus, with polynomially many vertices, the error of deterministic volume approximation is exponential in the worst case.
- Polytopes with such volume can be constructed: For $N = 2n$ use the crosspolytope, for $N = 4^n$ a 1-net in the dual S^{n-1}, and interpolate using a product.
- Ellipsoid: an affine image of B^n. John's lemma: Every n-dimensional convex body has inner and outer ellipsoids with ratio at most n, and a symmetric convex body admits the better ratio $\sqrt{n}$. The maximum-volume inscribed ellipsoid (which is unique) will do as the inner ellipsoid.

Chapter 14

- Measure concentration on S^{n-1}: For any set A occupying half of the sphere, almost all of S^{n-1} is at most $O(n^{-1/2})$ away from A. Quantitatively, $1 - \mathrm{P}[A_t] \le 2e^{-t^2 n/2}$.
- Similar concentration phenomena in many other high-dimensional spaces: Gaussian measure on $\mathbf{R}^n$, cube $\{0,1\}^n$, permutations, etc.
- Many concentration inequalities can be proved via isoperimetric inequalities. Isoperimetric inequality: Among all sets of given volume, the ball has the smallest volume of a t-neighborhood.
- Lévy's lemma: A 1-Lipschitz function f on S^{n-1} is within $O(n^{-1/2})$ of its median on most of S^{n-1}.
- Consequently (using η-nets), there is a high-dimensional subspace on which f is almost constant (use a random subspace).

- Normed spaces, norm induced by a symmetric convex body.
- For any n-dimensional symmetric convex polytope, $\log(f_0)\log(f_{n-1}) = \Omega(n)$ (many vertices or many facets).
- Dvoretsky's theorem: For every k and $\varepsilon > 0$ there exists n ($n = e^{O(k/\varepsilon^2)}$ suffices) such that any n-dimensional convex body has a k-dimensional $(1+\varepsilon)$-spherical section. In other words, any high-dimensional normed space has an almost Euclidean subspace.

Chapter 15

- Metric space; the distortion of a mapping between two metric spaces, D-embedding. Spaces ℓ_p^d and ℓ_p.
- Flattening lemma: Any n-point Euclidean metric space can be $(1+\varepsilon)$-embedded into ℓ_2^k, $k = O(\varepsilon^{-2}\log n)$ (project on a random k-dimensional subspace).
- Lower bound for D-embedding into a d-dimensional normed space: counting; take all subgraphs of a graph without short cycles and with many edges.
- The m-dimensional Hamming cube needs $\sqrt{m}$ distortion for embedding into ℓ_2 (short diagonals and induction).
- Edge expansion (conductance), second eigenvalue of the Laplacian matrix. Constant-degree expanders need $\Omega(\log n)$ distortion for embedding into ℓ_2 (tight). Method: Compare sums of squared distances over the edges and over all pairs, in the graph and in the target space.
- D-embeddability into ℓ_2 is polynomial-time decidable by semidefinite programming.
- All n-point spaces embed isometrically into ℓ_∞^n. For embeddings with smaller dimension, use distances to random subsets of suitable density as coordinates. A similar method yields $O(\log n)$-embedding into ℓ_2 (or any other ℓ_p).
- Example of algorithmic application: approximating the sparsest cut. Embed the graph metric into ℓ_1 with low distortion; this yields a cut pseudometric defining a sparse cut.