Markets and Money

Walrus (1874)
Prices be such that $\boxed{\text{demand} = \text{supply}}$

Fisher's linear case market

A set of goods (divisible) $|A| = n$
B buyers, $|B| = n'$
$i \in B$ has e_i money
$j \in A \Rightarrow$ there is b_j amount $[b_j = 1]$ ⊛
w_i: utility of $i \in B$ LINEAR:
given X_{ij} units of good $j \in A \Rightarrow$
$\boxed{i \text{ has utility } w_{ij} \cdot X_{ij}}$

Ass. ⊛ and w_{ij}, e_j integer
⊛ $\forall j \in A \ \exists i \in B: w_{ij} > 0$

Prices $p_1, \dots, p_n$ are market clearing prices if after each buyer is assigned an optimal basket of goods relative to the prices, there is no surplus or deficiency of any good.

Theorem Each good has a potential buyer, equilibrium exists and is unique.

Proof The Eisenberg-Gale convex program which can be solved in poly time (primal-dual alg) has a solution satisfying:

(i) $(\forall j)\ p_j \geq 0$

(ii) $p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$

(iii) $\forall i \in B\ \forall j_0 \in A\ \frac{u_{ij_0}}{p_{j_0}} \leq \frac{\sum_j u_{ij} x_{ij}}{e_i}$

(iv) $\forall i, j_0\ x_{ij_0} > 0 \Rightarrow \frac{u_{ij_0}}{p_{j_0}} = \frac{\sum_j u_{ij} x_{ij}}{e_i}$

• $(\forall j)$ there is $i : u_{ij} > 0$

(iii) $\Rightarrow p_j \geq \frac{e_i u_{ij}}{\sum_{j'} u_{ij'} x_{ij'}} > 0$

Hence prices of all goods positive, all goods sold by (ii)

(iii), (iv) $\Rightarrow$ if buyer i gets j $\Rightarrow$ j must be among the goods that give buyer i max utility per unit money spent at current prices.

— HENCE —

(*) i derives $\frac{u_{ij}}{p_j}$ utility per unit money spent on j

(*) $\boxed{\text{bang per buck } \alpha_i = \max_j \frac{u_{ij}}{p_j}}$

(iv) $\Leftrightarrow \forall i j_0\ \frac{e_i u_{ij_0} x_{ij_0}}{\sum_{j \in A} u_{ij} x_{ij}} = p_{j_0} x_{ij_0}$

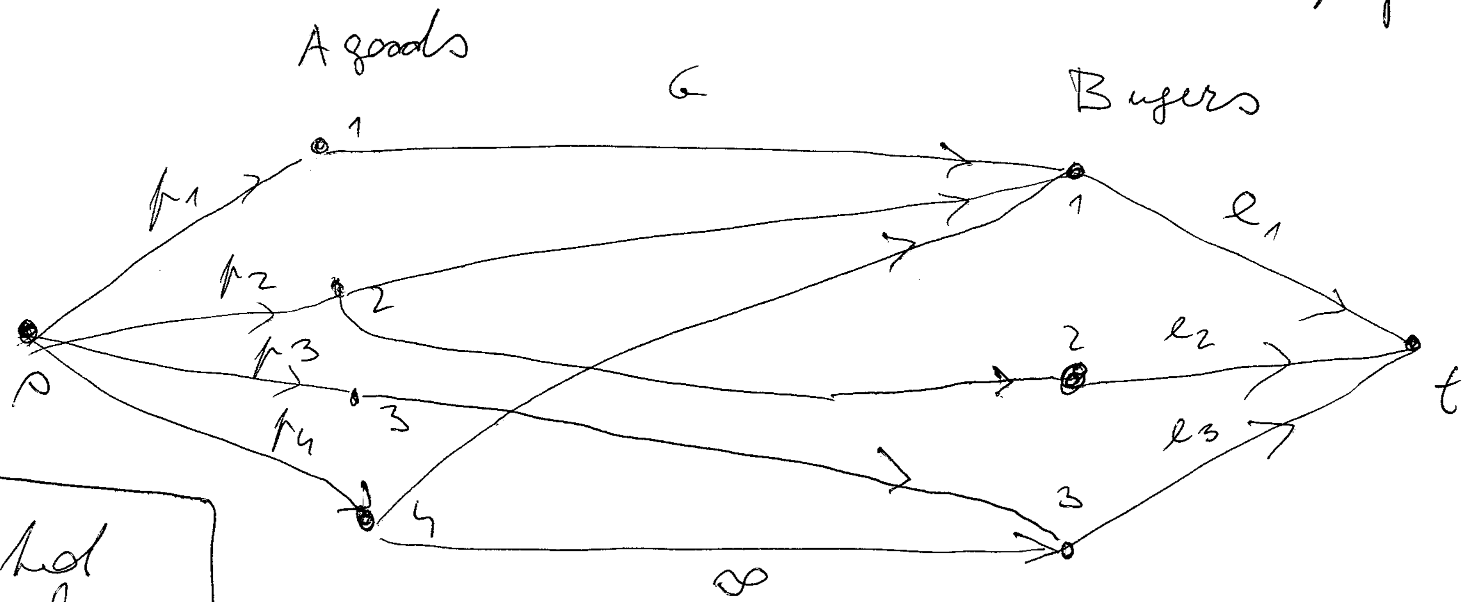
$(\forall i \in B)$
$\Rightarrow e_i = \sum_j p_j x_{ij}$ [all money spent]

$\boxtimes$

$\boxed{\text{uniqueness follows from the convex program}}$

Checking if prices $p = (p_1, \dots, p_n)$ are market clearing

G bip. graph (A, B), $(i, j) \in E(G)$ iff $\alpha_i = u_{ij}/p_j$. [bang per buck]



capacitated network

Max flow $\underline{\underline{f}}$

<u>Allocation of goods</u> : buyer i receives $\dfrac{f(j, i)}{p_j}$ units of good j.

$\boxed{\text{lemma}}$ p are market clearing prices iff cuts $(s, A \cup B \cup t)$, $(t, A \cup B \cup s)$ are min cuts; then, the allocation is equilibrium.

$\boxed{\text{linear case of Arrow-Debreu model}}$

A set of agents, G set of goods, $|A| = m$, $|G| = n$.

- Each agent i comes to MARKET with $e_i = (e_{i1} \dots e_{in})$ of goods.
- Ass. $\sum_{i=1}^{m} e_{ij} = 1$ [total amount of each good is 1]
- linear utilities: agent i has utility $\sum_{j=1}^{n} u_{ij} x_{ij}$
- FIND $p = (p_1, \dots, p_n)$ so that each agent sells, buys optimal basket and market clears [no deficiency or surplus]

Generalizes Fisher

$F(n \text{ goods}, m \text{ buyers}) \longrightarrow AD(n+1, m+1)$

AD:
- money is $(n+1)$st good
- buyers = agents, endowment $= e_i$
- $(m+1)$st agent endowment: all n goods
- First m agents have utilities for goods as in F, no utility for money
- $(m+1)$st agent has utility for money only.

Auction based algorithm

$\epsilon > 0$

fixed

$i \in A \Rightarrow a_i = \sum_{j=1}^{m} e_{ij}$

a_{min}, r_{max}, w_{min}, w_{max}

$\boxed{\text{Want}}$ Find p s.t. market clears and each agent gets basket of utility $\geq (1-\epsilon)^2 \cdot$ utility of opt. basket.

1. Initial prices 1 (min)
2. Give to $i \in A$ a_i $\boxed{\text{total surplus is } n}$
3. Iterations : terminates if price of good is raised by factor of $(1+\epsilon)$
4. each iteration partitioned into rounds.
5. In a round, agents considered in a fixed order. Agent i considered

no surplus money $\Downarrow$ next agent

finds i-th opt. good [bang per buck] at current prices, say j.

$\boxed{\text{OUTBID}}$ buying back good j from agents who have it at price p_j and selling to i at $p_j(1+\epsilon)$.

i-th surplus exhausted $\Downarrow$ next agent

no good j at price p_j $\Rightarrow$ raise price to $p_j(1+\epsilon)$ update agent's money

Total surplus money $\leq \epsilon \cdot a_{min} \Rightarrow$ END : unsold goods give arbitrarily.