Cubic bridgeless graphs and braces

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Abstract

There are many long-standing open problems on cubic bridgeless graphs, for instance, Jaeger's directed cycle double cover conjecture. On the other hand, many structural properties of braces have been recently discovered. In this work, we bijectively map the cubic bridgeless graphs to braces which we call the hexagon graphs, and explore the structure of hexagon graphs. We show that hexagon graphs are braces that can be generated from the ladder on 8 vertices using two types of McCuaig's augmentations. In addition, we present a reformulation of Jaeger's directed cycle double cover conjecture in the class of hexagon graphs.

1 Introduction

Jaeger's directed cycle double cover conjecture [1], usually known as DCDC conjecture, is broadly considered to be among the most important open problems in graph theory. A typical formulation asks whether every 2-connected graph admits a family of cycles such that one may prescribe an orientation on each cycle of the family in such a way that each edge e of the graph belongs to exactly two cycles and these cycles induce opposite orientations on e. In order to prove the DCDC conjecture, a wide variety of approaches have arisen [1, 10], among them, the topological approach. The topological approach claims that the DCDC conjecture is equivalent to the statement that every cubic bridgeless graph admits an embedding in a closed orientable surface such that every edge belongs to exactly two distinct face boundaries defined by the embedding; that is, with no dual loop.

In this work, we formulate the DCDC conjecture as a problem of existence of special perfect matchings in a class of graphs that we call hexagon graphs. Initially, our motivation for the formulation of the DCDC conjecture on hexagons are critical embeddings [4, 8], that in particular are embeddings with no dual loop.

The main goal of this work is to discuss recent progress on the study of the structure of hexagon graphs. The class of hexagon graphs of cubic bridgeless graphs turns out to be a subclass of braces.

*Partially supported by CONICYT: FONDECYT/POSTDOCTORADO 3150673, Núcleo Milenio Información y Coordinación en Redes ICM/FIC RC130003, Chile, FAPESP (Proc. 2013/03447-6) and CNPq (Proc. 456792/2014-7), Brazil.
†Partially supported by the German Research Foundation (KA 2748/2-1 and KA 2748/3-1).
‡Partially supported by the Czech Science Foundation under the contract number P202-13-21988S.

The class of braces, along with bricks, are a fundamental class of graphs in matching theory, mainly because they are building blocks of a perfect matching decomposition procedure; namely of the tight cut decomposition procedure [5]. In [7], McCuaig introduced a method for generating all braces starting from a large base set of graphs and recursively making use of 4 distinct types of operations. In this paper, we show that hexagon graphs are braces that can be generated from the ladder on 8 vertices using 2 types of McCuaig's operations.

In the following, we make precise the notions discussed above and formally state our main result.

1.1 Hexagon graphs

Hexagon graphs are the main ingredient and the center of attention of this work. In this section, we define the class of hexagon graphs, look over some of its fundamental properties and formulate the DCDC conjecture as a question about this new class of graphs. Despite our original motivation for this new formulation of the DCDC conjecture are critical embeddings, in this work we do not introduce this notion, and we present the details and proofs regarding the formulation using rotation systems of graphs, a well known and convenient combinatorial representation of embeddings on closed orientable surfaces [9, §3.2]. The advantage of using rotation systems is that we avoid topological arguments and present the equivalence to the DCDC conjecture in a purely combinatorial way.

We refer to the complete bipartite graph $K_{3,3}$ as a *hexagon* and say that a bipartite graph H has a hexagon h if h is a subgraph of H. For a graph G and a vertex v of G, let $N_G(v)$ denote the set of neighbors of v in G.

Definition 1 (Hexagon Graphs)**.** *Let G be a cubic graph with vertex set V and edge set E. A hexagon graph of G is a graph H obtained from G following the next rules:*

1. *We replace each vertex v in V by a hexagon h_v of H so that for every pair u, $v \in V$, if $u \neq v$, then h_u and h_v are vertex disjoint. Moreover, $V(H) = \{V(h_v) : v \in V\}$.*

2. *For each vertex $v \in V$, let $\{v_i : i \in \mathbb{Z}_6\}$ denote the vertex set of h_v and $\{v_i v_{i+1}, v_i v_{i+3} : i \in \mathbb{Z}_6\}$ its edge set. With each neighbor u of v in G, we associate an index $i_{v(u)}$ from the set $\{0, 1, 2\} \subset \mathbb{Z}_6$ so that if $N_G(v) = \{u, w, z\}$, then $i_{v(u)}$, $i_{v(w)}$, $i_{v(z)}$ are pairwise distinct.*

3. *See Figure 1. Let $X = \cup_{v \in V}\{v_{2i} : i \in \mathbb{Z}_6\}$ and $Y = \cup_{v \in V}\{v_{2i+1} : i \in \mathbb{Z}_6\}$. We replace each edge uv in E by two vertex disjoint edges e_{uv}, e'_{uv} so that if both $v_{i_{v(u)}}$, $u_{i_{u(v)}}$ belong to either X or Y, then $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)}+3}$, $e'_{uv} = v_{i_{v(u)}+3} u_{i_{u(v)}}$. Otherwise, $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)}}$, $e'_{uv} = v_{i_{v(u)}+3} u_{i_{u(v)}+3}$. Moreover, $E(H) = \{E(h_v) : v \in V\} \cup \{e_{uv}, e'_{uv} : uv \in E\}$.*

We say that h_v is the hexagon of H associated with the vertex v of G and that $\{h_v : v \in V\}$ is the *set of hexagons of H*. For $uv \in E$, we say that h_u and h_v are *hexagon-neighbors* in H. We shall refer to the set of edges $\bigcup_{v \in V}\{v_i v_{i+3} : i \in \mathbb{Z}_6\}$ as the set of *red edges of H*, to the set of edges $\{e_{uv}, e'_{uv} : uv \in E\}$ as the set of *white edges of H* and finally, to the set of edges $\bigcup_{v \in V}\{v_i v_{i+1} : i \in \mathbb{Z}_6\}$ as the set of *blue edges of H* (see Figure 1). Moreover, we shall say that a perfect matching of H containing only blue edges is a *blue perfect matching.*

Observation 1. *Hexagon graphs of cubic graphs are bipartite.*

Proof. Let H be a hexagon graph of a cubic bridgeless graph. Let X, Y be the sets defined in Definition 1, item *3.* Note that $\{X, Y\}$ is a partition of $V(H)$ and that there are no edges connecting vertices of the same partition class. □

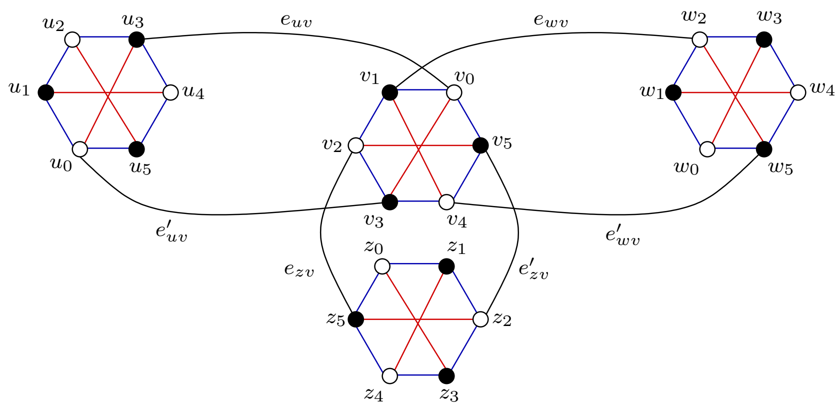


Figure 1: Local representation of the hexagon-neighborhood of a hexagon h_v in a hexagon graph H of a cubic graph G. The hexagon h_v is associated with vertex v, where $N_G(v) = \{u, w, z\}$. Red edges are depicted as red lines, blue edges are depicted as blue lines and white edges as black lines. The set X is represented by filled-in white vertices and the set Y by filled-in black vertices. Moreover, $i_{v(u)} = 0$, $i_{v(w)} = 1$, $i_{v(z)} = 2$, $i_{u(v)} = 0$, $i_{w(v)} = 2$ and $i_{z(v)} = 2$.

The following observation is straightforward.

Observation 2. *Let G be a cubic graph and H be a hexagon graph of G. The following properties hold.*

1. *H is a 4-regular graph.*
2. *No white edge of H connects two vertices of the same blue hexagon.*
3. *Both, the set of red edges of H and the set of white edges of H form a perfect matching of H.*
4. *Let $|V(G)|$ denote the cardinality of $V(G)$. There are $2^{|V(G)|}$ distinct blue perfect matchings.*

In the next statement we see that for each cubic bridgeless graph G, there exists a unique (up to isomorphism) hexagon graph. Note that the existence of an hexagon graph for each cubic graph is trivial since the choice of the indices $i_{v(u)}$, $i_{v(w)}$, $i_{v(z)}$ in Definition 1 for each vertex v is pairwise independent. In other words, there is an injective map from the set of all cubic bridgeless graphs to a special subset of braces (see also Theorem 8).

Proposition 1. *Let H and H' be hexagon graphs of the cubic graphs G and G', respectively. Then G and G' are isomorphic if and only if H and H' are isomorphic.*

Proposition 1 follows directly from Lemma 2.

Lemma 2. *Let H and H' be hexagon graphs of a cubic bridgeless graph. Let $\{h_1, \dots h_k\}$ and $\{h'_1, \dots h'_k\}$ be the set of hexagons of H and H', respectively. Then $\{h_1, \dots h_k\} = \{h'_1, \dots h'_k\}$.*

Proof. For the sake of contradiction let us assume that $\{h_1, \dots h_k\} \neq \{h'_1, \dots h'_k\}$. Therefore, there exists a hexagon, say $h \in \{h_1, \dots h_k\}$ such that h is not in $\{h'_1, \dots h'_k\}$. Let us suppose that $h = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ induces the local configuration depicted in Figure 1. By the assumption, we know that there exists an edge, say e, in the set of (red and blue) edges spanned by h (that is, in the hexagon spanned by h) that is a white edge with respect to the set of hexagons $\{h'_1, \dots h'_k\}$. By the symmetry of the hexagon, we can assume that e is a blue edge, say $e := v_1v_0$. Let h'_i and h'_j, with $i, j \in \{1, \dots, k\}$, be the hexagons of $\{h'_1, \dots h'_k\}$ such that $v_0 \in h'_i$ and $v_1 \in h'_j$; thus, $i \neq j$. It implies that $u_3, v_5, v_3 \in h'_i$ and $w_2, v_2, v_4 \in h'_j$ (see Figure 1). Then, v_3 has two neighbours in h'_j, namely, v_2 and v_4, a contradiction. $\square$

Rotation systems and embeddings without dual loops

Recall that our goal in this section is to reformulate the following statement: every cubic bridgeless graph admits an embedding on a closed orientable surface without dual loops. For this purpose, we now introduce a combinatorial representation of embedding of graphs on closed orientable surfaces; namely *rotation systems*.

Let G be a graph. For each $v \in V(G)$, let π_v be a cyclic permutation of the edges incident with v. A collection $\pi = \{\pi_v : v \in V(G)\}$ is called a *rotation system* of G. The proof of the following statement can be found in [9, §3.2].

Theorem 3. *Let π be a rotation system of a graph G. Then π encodes an embedding of G on a closed orientable surfaces with set of face boundaries*

$$\{e_1 e_2 \cdots e_k : e_i = v^i v^{i+1} \in E(G),\ \pi_{v^{i+1}}(e_i) = e_{i+1},\ e_{k+1} = e_1 \text{ and } k \text{ minimal}\}. \tag{1}$$

Moreover, the converse holds. That is, every embedding of G on a closed orientable surface defines a rotation system π of G where the set of face boundaries is given by the set described in (1).

In Theorem 4, we state that blue perfect matchings of hexagon graphs of a cubic graph G define embeddings of G on closed orientable surfaces with distinguished set of face boundaries, and vice versa. The proof is based on a natural bijection between blue perfect matchings and rotation systems. We first need to make an observation.

Observation 3. *Let M be a blue perfect matching of H and let W be the set of white edges of H. Each cycle C in $M \cup W$ induces a subgraph in G defined by the set of edges $\{uv \in E(G) : e_{uv} \in C \text{ or } e'_{uv} \in C\}$.*

Theorem 4. *Let G be a cubic graph, H be the hexagon graph of G and W be the set of white edges of H. Each blue perfect matching M of H encodes an embedding of G on a closed orientable surface with a set of face boundaries, the set of subgraphs of G induced by the cycles in $M \cup W$. Moreover, the converse holds. That is, each embedding of G on a closed orientable surface defines a blue perfect matching M of H, where the set of subgraphs of G induced by all cycles in $M \cup W$ coincides with the set of face boundaries of the embedding.*

Proof. It suffices to prove that there is a bijective function f from the set of blue perfect matchings of H to the set of rotation systems of G such that for every blue perfect matching M of H, the set of subgraphs of G induced by the cycles in $M \cup W$ equals the set of subgraphs described in (1) defined by the rotation system $f(M) = \pi$.

Let $v \in V(G)$, $N_G(v) = \{u, w, z\}$, and without loss of generality (by Proposition 1) we assume that $i_{v(u)} = 0$, $i_{v(w)} = 1$ and $i_{v(z)} = 2$. Let M be a blue perfect matching of H. The restriction of M to h_v is either $\{v_0v_1, v_2v_3, v_4v_5\}$ or $\{v_1v_2, v_3v_4, v_5v_0\}$. If the restriction is $\{v_0v_1, v_2v_3, v_4v_5\}$, then the cyclic permutation of the edges incident with v in the rotation system $f(M) = \pi$ of G is $\pi_v = (uv\ wv\ zv)$. Otherwise, the cyclic permutation is given by $\pi_v = (uv\ zv\ wv)$. It is a routine to check that f is the desired bijection. □

The following result is crucial for our approach.

Proposition 5. *Let G be a cubic graph, H be the hexagon graph of G, M be a blue perfect matching of H and W be the set of white edges of H. The embedding of G encoded by M has a dual loop if and only if there is a cycle in $M \cup W$ that contains the end vertices of a red edge.*

Proof. An embedding of G has a dual loop if and only if there is an edge $uv \in E(G)$ that belongs to exactly one face boundary, say C'. The face boundary C' is a subgraph of G induced by a cycle C of $M \cup W$. We have C' is the only subgraph induced by a cycle of $M \cup W$ that contains uv if and only if e_{uv} and e'_{uv} belong to C. The lemma follows. □

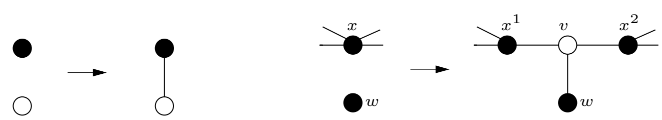
Motivated by Proposition 5, we shall say that a blue perfect matching M is *safe* if no cycle of $M \cup W$ contains the end vertices of a red edge. In Corollary 6 we establish the formulation of the DCDC Conjecture on hexagon graphs. Note that the result of Corollary 6 follows directly from Theorem 4 and Proposition 5.

Corollary 6. *A cubic graph G has a directed cycle double cover if and only if its hexagon graph H admits a safe perfect matching.*

1.2 Braces

A *brace* is a simple (that is, no loops and no multiple edges), connected, bipartite graph on at least six vertices, and with a perfect matching such that for every pair of nonadjacent edges, there is a perfect matching containing the pair of edges. In [7], McCuaig presented a method for generating braces. He showed that all braces can be constructed from a base set using four operations. In the following we describe McCuaig's method for generating braces.

Let H be a bipartite graph and x be a vertex of H of degree at least 4. Let N_1, N_2 be a partition of $N_H(x)$ such that $|N_1|, |N_2| \geq 2$. Let $\{x^1, v, x^2\}$ be a set of vertices such that $\{x^1, v, x^2\} \cap V(H) = \emptyset$. The *expansion of x to x^1vx^2*, or briefly an *expansion of x* is the operation composed of the following three steps: (i) delete x, (ii) add the new path x^1vx^2, and (3) connect every vertex of N_1 (N_2, respectively) to the vertex x^1 (x^2, respectively). For $i \in \{1, 2\}$, we say that N_i is the partition associated with x^i. Note that if H' is a graph obtained from H by the expansion of a vertex, then H' is also bipartite.
Augmentations. If H' is a bipartite graph obtained from H by adding a new edge, then we say that H' is obtained from H by a *type-1 augmentation*. Let x and w be two vertices in the same partition class of H such that x has degree at least 4. If H' is obtained from H expanding x to x^1vx^2 and adding the new edge vw, then we say that H' is obtained from H by a *type-2 augmentation*. Let x and y be two vertices of H of distinct partition classes such that $d_H(x), d_H(y) \geq 4$. Let H' be the bipartite graph obtained from H by expanding x and y to x^1vx^2 and y^1uy^2 respectively, and adding the new edge vu. If x and y are not connected in H, the operation for obtaining H' from H is called a *type-3 augmentation*, otherwise it is called a *type-4 augmentation*.

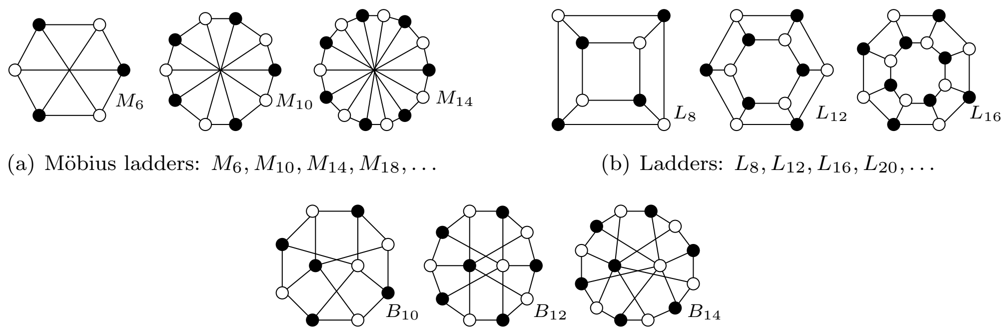


(a) type-1 augmentation (b) type-2 augmentation

Figure 2: Simple augmentations

If H' is obtained from H by a type i augmentation for some $i \in \{1, 2, 3, 4\}$, then we say that H' is obtained from H by an *augmentation*. If $i \in \{1, 2\}$, then we say that H' is obtained from H by a *simple augmentation* (see Figure 2).

Let $\mathcal{B}$ be the infinite set consisting of all bipartite Möbius ladders, ladders and biwheels (see Figure 3).



(a) Möbius ladders: $M_6, M_{10}, M_{14}, M_{18}, \ldots$

(b) Ladders: $L_8, L_{12}, L_{16}, L_{20}, \ldots$

(c) Biwheels: $B_{10}, B_{12}, B_{14}, B_{16}, \ldots$

Figure 3: The base set $\mathcal{B}$.

Theorem 7 (McCuaig, 1998)**.** *Let H be a bipartite graph. Then H is a brace if and only if there exists a sequence $H_0, H_1, \ldots, H_k$ of bipartite graphs such that $H_0 \in \mathcal{B}$, H_i may be obtained from H_{i-1} by an augmentation for each $i \in \{1, \ldots, k\}$ and $H_k = H$.*

1.3 Main results

The main results of this paper are the following.

Theorem 8. *Let G be a cubic graph. Then the hexagon graph H of G is a brace if and only if G is bridgeless.*

Proof. Let B, W, and R denote the set of blue, white, and red edges, respectively. Moreover, a blue edge is denoted by b, a white edge by w, and a red edge by r. Each pair of disjoint edges, $\{b, b'\}$, $\{r, r'\}$, or $\{b, r\}$, can be simply extended to a perfect matching of H.

We note that each component of $W \cup R$ is a cycle on four vertices, a *square*. Let w, w' be a pair of disjoint white edges. The edges w, w' belong to the same square of $W \cup R$, or to two different squares of $W \cup R$. In either case w, w' can be naturally extended to a perfect matching of H. Similarly, each edge of a pair w, r of disjoint white and red edges belongs to different squares of $W \cup R$, and therefore it can be completed into a perfect matching of H.

Finally we consider a pair b, w of disjoint white and blue edges. If the hexagon with b does not contain an end vertex of w, then it is not difficult to extend b, w to a perfect matching of H. Hence, let h_u be the hexagon that contains b and an end vertex of w, and let h_v be the hexagon that contains the other end vertex of w. Let $b = u_i u_{i+1}$, $w = u_k v_j$, where $i, j, k \in \mathbb{Z}_6$.

If $k \notin \{i+3, i+4\}$, then b, w can be completed into a perfect matching of H that contains the edges b, w, and $u_{i+3}u_{i+4}$.

Hence, without loss of generality we can assume that $k = i + 3$. Let $e_{uv} = u_i v_{j+3}$ and $e_{uz} = u_{i+1} z_l$ (notation as in (3) of Definition 1), where z is the neighbor of v in G such that the white edge with an end vertex u_{i+1} has an end vertex in h_z, and $l \in \mathbb{Z}_6$. Given that in G, edges uv, uz have a common end vertex u represented by hexagon h_u, edge $b = u_i u_{i+1}$ can be seen as *the transition* between uv, uz, while $u_k u_{k+1}$ can be seen as this transition reversed.

Now let G be bridgeless. We observe that two adjacent edges in a cubic bridgeless graph belong to a common cycle. Let C be such a cycle for uv, uz.

The two possible orientations of C correspond to two disjoint cycles C_b, C_w in H, where $b \in C_b$ and $w \in C_w$; they contain the transition and transition reversed (between uv, uz), respectively. Let M_b be the perfect matching of C_b consisting of all blue edges and M_w be the perfect matching of C_w consisting of all white edges. In particular, $b \in M_b$ and $w \in M_w$. Since each hexagon of H is intersected by $C_b \cup C_w$ either in a pair of disjoint blue edges, or in the empty set, $M_b \cup M_w$ can be extended to a perfect matching of H.

On the other hand, if G has a bridge $e = \{u, v\}$, then let V_1 be the component of $G - e$ containing u. Any perfect matching of G extending b, w must induce a perfect matching of $\cup_{x \in V_1} h_x \setminus \{u_{i+3}\}$, but this set consists of an odd number of vertices and thus no perfect matching containing b, w can exist. □

Theorem 9. *Let G be a cubic bridgeless graph and L_8 denote the ladder on 8 vertices. There is a sequence $H_0, H_1, \ldots, H_k$ of bipartite graphs such that $H_0 = L_8$, H_i can be obtained from H_{i-1} by a simple augmentation for each $i \in \{1, \ldots, k\}$ and H_k is the hexagon graph of G.*

The crucial ingredients in the proof of Theorem 9 are odd ear decompositions of cubic bridgeless graphs. We now give a rough sketch of the proof. Let G be a cubic bridgeless graph, H be its hexagon graph, and $(G_0, G_i, P_i)^l$ be an odd ear decomposition of G (see Subsection 3.1). With each intermediate subgraph G_i of the odd ear decomposition of G we associate an auxiliary graph H'_i. In particular, with (the cycle) G_0 we associate the ladder L_8. For each $i \in \{1, \ldots, l\}$, the auxiliary graph H'_i contains the hexagons h_v of H such that v has degree 3 in G_i. Hence, H'_l contains all hexagons of H and indeed (by construction) it turns out to be isomorphic to H. The proof is based on the fact that for each $i \in \{1, \ldots, l\}$, it is possible to generate H'_i from H'_{i-1} by a sequence of simple augmentations.

The rest of the paper is devoted to prove Theorem 9. The proof of Theorem 9 is divided into two parts. The first part is the generation of hexagon graphs from square graphs and the second is the construction of square graphs from the ladder on 8 vertices. In Section 2, we introduce the concept of square graphs and prove that hexagon graphs can be obtained from square graphs by a short sequence of simple augmentations. Section 3 and Section 4 focus on the construction of square graphs.

2 Square graphs

A *square* is a complete bipartite graph on 4 vertices, namely $K_{2,2}$. We say that a bipartite graph has a square s if it contains s as a subgraph. Next we define *square graphs*.

Definition 2 (Square graphs)**.** *Let G be a cubic bridgeless graph with vertex set V and edge set E. Let M be a perfect matching of G. An M-square graph of G is a bipartite graph Q with neither loops nor multiple edges satisfying the following properties:*

1. *For each vertex v in V, the graph Q has a square s_v. If $u, v \in V$ are such that $u \neq v$, then s_v and s_u are vertex disjoint subgraphs of Q. Moreover, $V(Q) = \{V(v) : v \in V\}$.*
2. *The set of edges of Q is given by*

$$E(Q) = \{E(s_v) : v \in V\} \cup \{\boldsymbol{uv} : uv \in E\},$$

where $\{\boldsymbol{uv} : uv \in E\}$ is defined such that the following conditions hold:

   (a) *For each edge $uv \in E$, there are edges e_u in $E(s_u)$ and e_v in $E(s_v)$ such that the subgraph of Q induced by the set of edges $\{e_u, e_v\} \cup \boldsymbol{uv}$ is isomorphic to $K_{2,2}$. In particular, $|\boldsymbol{uv}| = 2$. The edges e_u and e_v are called* the supporting edges of $\boldsymbol{uv}$ *in s_u and s_v, respectively.*

   (b) *Let $v \in V$ and $N_G(v) = \{u, w, z\}$. If $uv \in M$, then the supporting edges of $\boldsymbol{wv}$ and $\boldsymbol{zv}$ in s_v are vertex disjoint.*

We say that s_v is the *square associated* with vertex v and that $\{s_v : v \in V\}$ is the set of squares of Q. For each $uv \in E$, if $uv \in M$, then we say that (s_u, s_v) is a *pair of matched squares of* Q. Moreover, the subset of edges $\boldsymbol{uv}$ is called *the projection of* uv *in* Q. We usually denote by $\{v_i : i \in \mathbb{Z}_4\}$ the vertex set of the square s_v and by $\{v_i v_{i+1} : i \in \mathbb{Z}_4\}$ its edge set.

Note that the graph obtained by contracting each square of Q to a single point and then by deleting multiple edges is precisely G. The following is a natural observation about square graphs.

Observation 4. *For every connected component C of $G - M$ (C is a cycle since G is cubic), there exists a ladder L on $4 \cdot |C|$ vertices in the set of connected components of $Q - \{\boldsymbol{e} : e \in M\}$ such that v is a vertex of C if and only if s_v is a square of L.*

In Lemma 10, we state that hexagon graphs can be generated from square graphs using simple augmentations.

Lemma 10. *Let G be a cubic bridgeless graph, M be a perfect matching of G and Q be an M-square graph of G. Then there is a sequence of bipartite graphs $H_0, H_1, \ldots, H_l$ such that $H_0 = Q$, H_i may be obtained from H_{i-1} by a simple augmentation for each $i \in \{1, \ldots, l\}$ and H_l is the hexagon graph of G.*

Proof. We first describe an operation composed of a sequence of simple augmentations which we apply to each pair of matched squares in order to generate a pair of hexagon-neighbors; we shall call this operation a *double augmentation.* Let (s_u, s_v) be a pair of matched squares of Q. By definition, all distinct configurations of the supporting edges of $\boldsymbol{uv}$ in s_u and s_v, respectively, are the ones depicted in Figure 4.

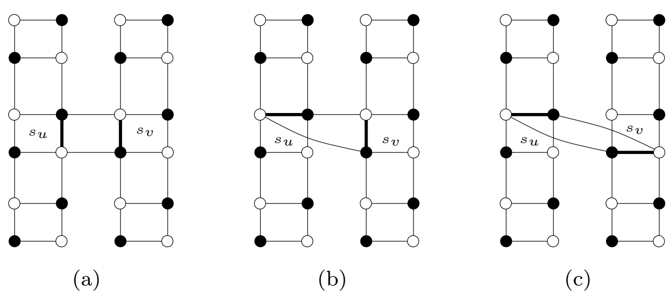
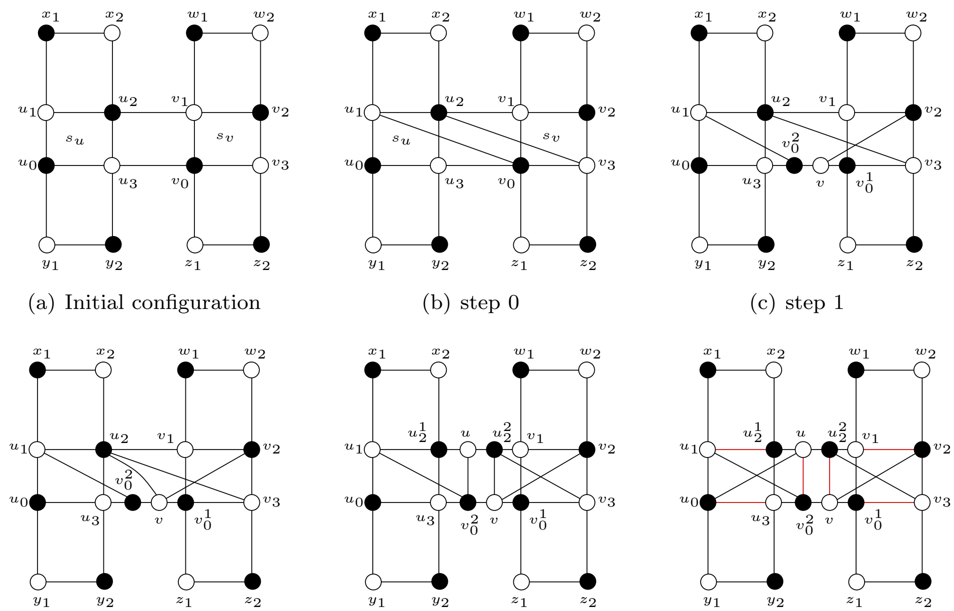


Figure 4: Possible locations of the supporting edges of $\boldsymbol{uv}$ in s_u and s_v for a pair (s_u, s_v) of matched squares of Q. Supporting edges are depicted by thick lines.

We assume that the supporting edges of $\boldsymbol{uv}$ for the pair (s_u, s_v) are configured as in Figure 4(a). Consider the vertex labeling depicted in Figure 5(a). Next, we describe the aforementioned operation with input the pair (s_u, s_v).

Double augmentation on (s_u, s_v): (see Figure 5) [step 0:] addition of the two new edges u_1v_0 and u_2v_3. [step 1:] expansion of v_0 to $v_0^1 v v_0^2$ in such a way that the partition associated with v_0^2 is $\{u_1, u_3\}$ and with v_0^1 is $\{v_1, v_3, z_1\}$ and addition of the new edge vv_2. [step 2:] addition of the new edge vu_2. [step 3:] expansion of u_2 to $u_2^1 u u_2^2$ in such a way that the partition associated with u_2^1 is $\{u_1, x_2, u_3\}$ and with u_2^2 is $\{v_1, v, v_3\}$ and addition of the new edge uv_0^2. [step 4:] addition of the new edge uu_0. We observe that in steps 1 and 3 respectively, expansion of v_0 and expansion of u_2 respectively are allowed given that the degrees are 5 and 6 respectively; recall that degree at least 4 is required for expansion; see Subsection 1.2.

In case that the supporting edges of $\boldsymbol{uv}$ for the pair (s_u, s_v) are configured as in Figure 4(b) or as in Figure 4(c) respectively (set the same vertex labeling), if we replace the edges added at the step 0 of the double augmentation described above by u_2v_3, u_3v_0 and u_2v_1, u_3v_0 respectively, then the local configuration obtained is the one depicted in Figure 5(b). Therefore, if we continue applying steps 1, 2, 3 and 4 as before we obtain the local configuration depicted in Figure 5(f).



(a) Initial configuration (b) step 0 (c) step 1

(d) step 2 (e) step 3 (f) step 4

Figure 5: Double Augmentation on (s_u, s_v). In subfigure (f), red edges are depicted by red lines.

We claim that the graph obtained from Q by performing a double augmentation on every pair of matched squares is a hexagon graph of G. The disjoint subsets of vertices $\{u_0, u_3, u_1, u_2^1, u, v_0^2\}$ and $\{v_1, v_2, v_3, v_0^1, v, u_2^2\}$ induce hexagons. Let h_u and h_v denote them respectively. The claim follows by setting $\{u_0u_3, u_1u_2^1, uv_0^2\}$ and $\{v_1v_2, v_3v_0^1, vu_2^2\}$ to be the subsets of red edges in h_u and h_v, respectively (see Figure 5(f)).

To conclude, since steps 0, 2 and 4 correspond to type-1 augmentations, and steps 1 and 3 correspond to type-2 augmentations, we have that a double augmentation on a pair of matching related squares is composed of a sequence of simple augmentations. □

3 Construction of square graphs

In order to prove Theorem 9, by Lemma 10 it suffices to show that we can construct an M-square graph of G, for some perfect matching M of G, from the ladder on 8 vertices using simple augmentations. In this section we develop a method to construct square graphs following an ear decomposition of the underlying cubic bridgeless graph G and using simple augmentations.

3.1 Odd ear decomposition of a cubic bridgeless graph

Let G be a graph. We say that a path, or a cycle of G, is *even* (*odd* respectively) if it has an even (odd respectively) number of edges. An *odd ear decomposition* of G, denoted by $(G_0, G_i, P_i)^l$, consists of a sequence of subgraphs $G_0, G_1, \ldots, G_l$ and a sequence of odd paths $P_1, \ldots, P_l$ of G such that G_0 is an even cycle of G, $G_l = G$ and for each $i \in \{1, \ldots, l\}$ the subgraph G_i is obtained from G_{i-1} joining two vertices α_i and β_i in $V(G_{i-1})$ by a path P_i, where P_i is such that $V(P_i) \cap V(G_{i-1}) = \{\alpha_i, \beta_i\}$ and

$E(P_i) \cap E(G_{i-1}) = \emptyset$. It is folklore that every edge of a cubic bridgeless graph is contained in a perfect matching and hence, the class of cubic bridgeless graph is a subclass of the class of 1-extendable graphs. In addition, every 1-extendable graph admits an odd ear decomposition [6, §5.4].

Let G be a cubic bridgeless graph and $(G_0, G_i, P_i)^l$ be an odd ear decomposition of G. We say that a perfect matching M of G is *absolute* in $(G_0, G_i, P_i)^l$ if the restriction of M to $E(G_i)$ is a perfect matching of G_i for every $i \in \{0, 1, \ldots, l\}$. The next observation is straightforward.

Observation 5. *For every odd ear decomposition $(G_0, G_i, P_i)^l$ of a cubic bridgeless graph G, there exists a perfect matching M of G that is absolute in $(G_0, G_i, P_i)^l$.*

In the rest of the paper, we deal only with perfect matchings that are absolute in a given odd ear decomposition $(G_0, G_i, P_i)^l$. Let $i \in \{1, \ldots, l\}$ and let $V_j(G_i)$ denote the subset of vertices of $V(G_i)$ that have degree j in G_i for each $j \in \{2, 3\}$. Let $u, v \in V_3(G_i)$ and P be a path of G_i with end vertices u, v such that $V(P) \cap V_3(G_i) = \{u, v\}$. In other words, every inner vertex of P belongs to $V_2(G_i)$. We say that P is a (u, v)-path of G_i and usually denote P by $p(u, v)$. Note that there may exist multiple (u, v)-paths. We shall denote by $\mathcal{P}(G_i)$ the set of all (u, v)-paths for all u, v in $V_3(G_i)$.

We note that if v is a vertex in $V_3(G_i)$, then there are three (not necessarily distinct) vertices x, y, z in $\mathcal{V}(G_i)$, such that $p(x, v)$, $p(y, v)$, $p(z, v) \in \mathcal{P}(G_i)$. We say that the set $\{x, y, z\}$ is the *set of pseudo-neighbors* of v in G_i.

Observe that if M is a perfect matching of G and $vw \in M$, then there is a unique path $P \in \{p(x, v), p(y, v), p(z, v)\}$ such that $vw \in E(P)$. We refer to P as the *matching-path* of v in G_i (with respect to M). If vw is not in $E(P)$, then P is called a *cycle-path* of v in G_i. Note that a path $p(u, v)$ in $\mathcal{P}(G_i)$ could be both, a matching-path of v and a cycle-path of u. However, since M is a perfect matching that is absolute in $(G_0, G_i, P_i)^l$, the path $P_i = p(\alpha_i, \beta_i) \in \mathcal{P}(G_i)$ is always a cycle-path of both α_i and β_i in G_i (see Figure 9(a)).

In Subsection 3.2, we generalize the definition of square graphs of a cubic graph G to the intermediate graphs $G_0, G_1, \ldots, G_l$ associated with an odd ear decomposition of G.

3.2 Ear square graphs

In this section and in the rest of the paper, G is a cubic bridgeless graph, $(G_0, G_i, P_i)^l$ is an odd ear decomposition of G and M is a perfect matching of G that is absolute in $(G_0, G_i, P_i)^l$.

Definition 3 (Ear square graphs)**.** *For each $i \in \{1, \ldots, l\}$, a (G_i, M)-*ear square graph *is a bipartite graph Q_i with neither loops nor multiple edges that satisfies the following properties:*

1. *For each vertex v in $V_3(G_i)$, the graph Q_i has a square s_v. For every u,v in $V_3(G_i)$ with $u \neq v$, the squares s_v and s_u are vertex disjoint subgraphs of Q_i. Moreover, $V(Q_i) = \{V(s_v) : v \in V_3(G_i)\}$.*

2. *The set of edges of Q_i is given by*

$$\{E(s_v) : v \in V_3(G_i)\} \ \dot{\bigcup_{p(u,v) \in \mathcal{P}(G_i)}} \ \mathbf{p}(u, v)$$

   where $\{\boldsymbol{p}(u, v) : p(u, v) \in \mathcal{P}(G_i)\}$ is defined such that the following conditions hold:

   (a) For each $p(u, v) \in \mathcal{P}(G_i)$, we have $|\boldsymbol{p}(u, v)| = 2$, and there are edges e_u in $E(s_u)$, e_v in $E(s_v)$ such that the subgraph of Q_i induced by the set of edges $\{e_u, e_v\} \cup \boldsymbol{p}(u, v)$ is isomorphic to $K_{2,2}$. The edges e_u and e_v are called the supporting edges of $\mathbf{p}(u, v)$ *in s_u and s_v, respectively.*

(b) Let v be a vertex in $V_3(G_i)$ and $\{x, y, z\}$ be its set of pseudo-neighbors. If $p(x, v)$ is the matching-path of v in G_i, then the supporting edges of $\boldsymbol{p}(v, y)$ and $\boldsymbol{p}(v, z)$ in s_v are vertex disjoint (see Figure 6).

(c) Elements in $\{\boldsymbol{p}(u, v) : p(u, v) \in \mathcal{P}(G_i)\}$ are pairwise disjoint.

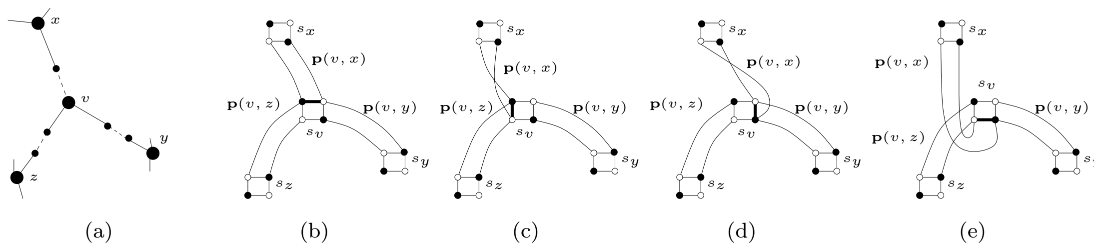


Figure 6: Local representation of G_i and a (G_i, M)-ear square graph of G_i. In subfigure (a), we depict a vertex $v \in V_3(G_i)$ with $x, y, z \in V_3(G_i)$ its pseudo-neighbors and $p(v, x)$ the matching-path of v in G_i. Dashed edges represent edges from M. In subfigures (b)-(e), we depict all the allowed locations of the supporting edge of $\mathbf{p}(v, x)$ in s_v in a (G_i, M)-ear square graph of G_i. In each subfigure the supporting edge is depicted by a thicker line.

For every $p(u, v) \in \mathcal{P}(G_i)$, the set $\mathbf{p}(u, v)$ is said to be its *projected (u, v)-path in Q_i*. If $p(u, v)$ is the matching-path of v in G_i, we say that $\mathbf{p}(u, v)$ is the *projected matching-path of s_v in Q_i*.

Since $V_3(G_l) = V(G)$, the following proposition follows from Definition 2 and Definition 3.

Observation 6. *A graph H is a (G_l, M)-ear square graph if and only if H is an M-square graph.*

In Lemma 11 we formalize the construction of square graphs using ear square graphs and simple augmentations. This lemma is proved in Section 4.

Lemma 11 (Construction of square graphs)**.** *Let G be a cubic bridgeless graph, $(G_0, G_i, P_i)^l$ be an odd ear decomposition of G and M be a perfect matching of G that is absolute in $(G_0, G_i, P_i)^l$. Let L_8 denote the ladder on 8 vertices (see Figure 3(b)). The following two properties hold.*

1. *A (G_1, M)-ear square graph Q_1 can be generated from L_8 using type-1 augmentations.*
2. *Let $i \in \{2, \dots, l\}$ and Q_{i-1} be a (G_{i-1}, M)-ear square graph. Then a (G_i, M)-ear square graph Q_i can be generated from Q_{i-1} using a sequence of simple augmentations.*

Note that Lemma 11 along with Observation 6 and Lemma 10 imply Theorem 9.

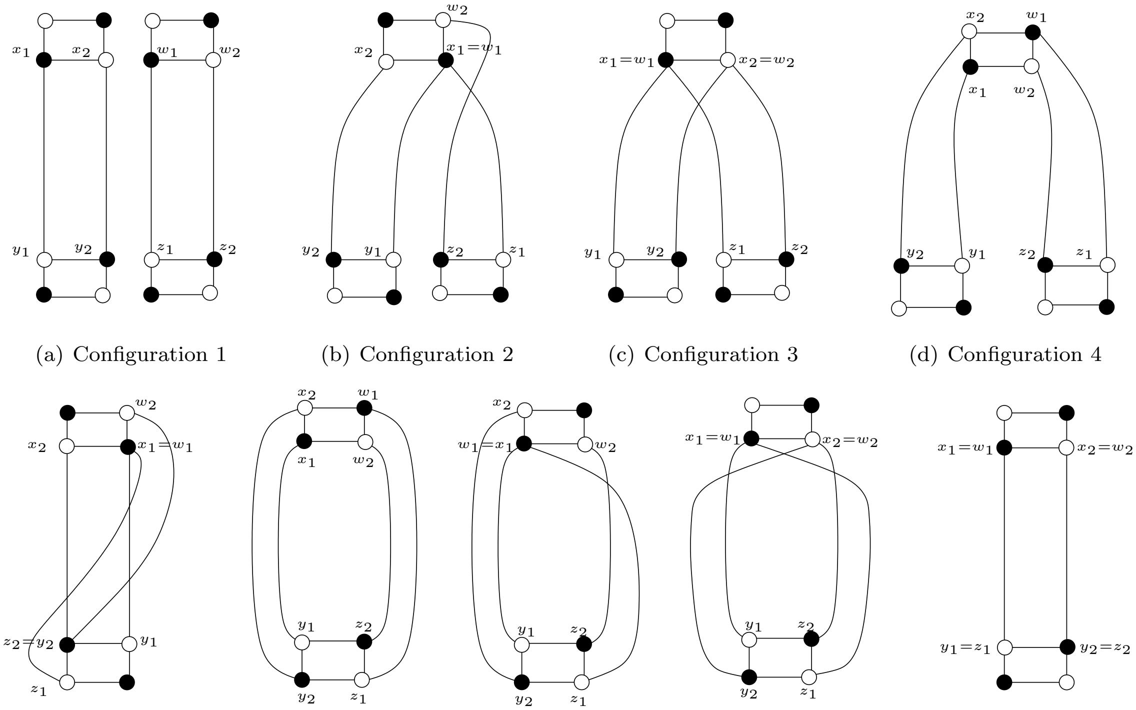
4 Proof of Lemma 11

In this section, G is a cubic bridgeless graph, $(G_0, G_i, P_i)^l$ is an odd ear decomposition of G and M is a perfect matching of G that is absolute in $(G_0, G_i, P_i)^l$. Let L_8 denote the ladder on 8 vertices. Moreover, for each $i \in \{1, \dots, l\}$, let Q_i denote a (G_i, M)-ear square graph.

In what follows we enunciate two natural properties about ear square graphs. The result of Proposition 12 follows directly from Definition 3.

Proposition 12. *For every $i \in \{1, \dots, l\}$, each square s_v in Q_i with $V(s_v) = \{v_j : j \in \mathbb{Z}_4\}$ is such that there exists a unique $j \in \mathbb{Z}_4$ such that $d_{v_j} = d_{v_{j+1}} = 4$ and $d_{v_{j+2}} = d_{v_{j+3}} = 3$.*

Proposition 13. *Let $p(x,y)$ and $p(w,z)$ be paths in $\mathcal{P}(G_i)$. Let $\boldsymbol{p}(x,y)$ be the projected path of $p(x,y)$ and $\boldsymbol{p}(w,z)$ be the projected path of $p(w,z)$ in Q_i. Then the subgraph S of Q_i with a set of edges $\boldsymbol{p}(x,y) \cup \boldsymbol{p}(w,z) \cup E(s_x \cup s_y \cup s_w \cup s_z)$ is isomorphic to one of the 9 graphs (configurations) depicted in Figure 7.*



(a) Configuration 1 (b) Configuration 2 (c) Configuration 3 (d) Configuration 4

(e) Configuration 5 (f) Configuration 6 (g) Configuration 7 (h) Configuration 8 (i) Configuration 9

Figure 7: In (a) is depicted the unique subgraph that arises when vertices $x, y, z, w \in V_3(G_i)$ are all distinct. From (b) to (d) the three possible subgraphs that arise when $|\{s_x, s_y, s_w, s_z\}| = 3$. Figures from (e) to (i) depict all the possible situations when $|\{s_x, s_y, s_w, s_z\}| = 2$.

Proof. We first suppose that $|\{x, y, w, z\}| = 4$. Then $x, y, z, w \in V_3(G_i)$ are all distinct and the squares s_x, s_y, s_w, s_z in Q_i are vertex disjoint. Therefore, S is isomorphic to the graph depicted in Figure 7(a).

We now suppose that $|\{x, y, w, z\}| = 3$. It means that the paths $p(x,y)$ and $p(w,y)$ have one common end vertex. Without loss of generality we suppose that $x = w$, and then, s_x, s_y and s_z are vertex disjoint. In the subgraph S three distinct situations depending on the location of the supporting edges e_x and e_w of $\mathbf{p}(x,y)$ and $\mathbf{p}(w,z)$ in s_x can arise:

a.1) either $|e_x \cap e_w| = 1$, or

a.2) $e_x = e_w$, or

a.3) $e_x \cap e_w = \emptyset$.

If situation a.1) holds, then S is isomorphic to configuration 2, see Figure 7(b). If situation a.2) holds, then S is isomorphic to configuration 3, see Figure 7(c), and if situation a.3) holds, then S is isomorphic to configuration 4, see Figure 7(d).

We finally suppose that $|\{x, y, w, z\}| = 2$. Without loss of generality we assume that $x = w$ and $y = z$. If $p(x, y) = p(w, z)$, then S is isomorphic to the graph depicted in Figure 7(i), this graph is called configuration 9. Otherwise, in the graph S several distinct situations depending on the location of the supporting edges e_x and e_w of $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ in s_x and of the supporting edges e_y and e_z of $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ in s_y may arise:

b.1) either $|e_x \cap e_w| = 1$ and $|e_y \cap e_z| = 1$, or

b.2) $e_x \cap e_w = \emptyset$ and $e_y \cap e_z = \emptyset$, or

b.3) $|e_x \cap e_w| = 1$ and $e_y \cap e_z = \emptyset$, or

b.4) $e_x = e_w$ and $e_y \cap e_z = \emptyset$, or

b.5) $|e_x \cap e_w| = 1$ and $e_y = e_z$, or $e_x = e_w$ and $e_y = e_z$.

If situation b.1), b.2), b.3) or b.4) holds, then S is isomorphic to configuration 5, 6, 7, or 8, respectively. Those configurations are depicted in Figure 7). Situations described in b.5) do not occur in Q_i given that Q_i does not have multiple edges. We clarify the last statement in the following paragraph.

If we suppose that $|e_x \cap e_w| = 1$ or $e_x = e_w$ then, there exists a vertex, without loss of generality we assume that such a vertex is $x_1 \in e_x \cap e_y$ such that x_1y_1 and x_1z_1 are edges of S. We recall that Q_i does not have multiple edges. Since $e_y = e_z$, we have $y_1 = z_1$ and S has a double edge, a contradiction. □

4.1 Generating ear square graphs

This section is devoted to prove Lemma 11.

Proof of Lemma 11, part *1*

We need to prove that we can generate a (G_1, M)-ear square graph from L_8 using type-1 augmentations (addition of new edges). We consider the vertex-labeling of L_8 depicted in Figure 8(a). Let $p(u, v)$, $p(x, y)$ and $p(w, z)$ be the only three paths in $\mathcal{P}(G_1)$, where $u = x = w$, $v = y = z$ and $P_1 = p(u, v)$. We have that G_1 satisfies one of the following properties:

1) either $p(x, y)$ is the matching-path of v and of u in G_1, or

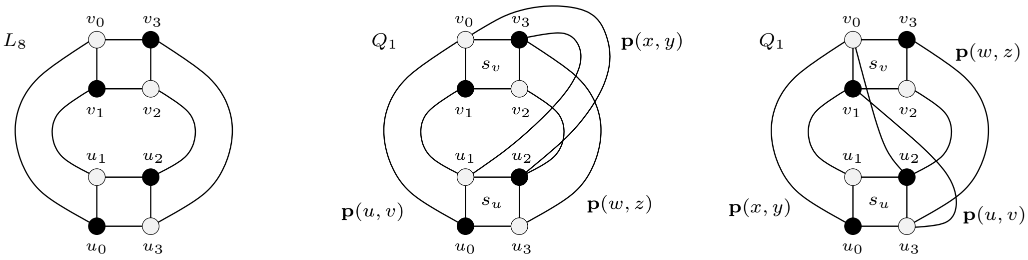
1′) $p(w, z)$ is the matching-path of v and of u in G_1, or

2) $p(x, y)$ is the matching-path of v in G_1 and $p(w, z)$ is the matching-path of u in G_1, or

2′) $p(w, z)$ is the matching-path of v in G_1 and $p(x, y)$ is the matching-path of u in G_1.

By symmetry, it suffices to prove that for each $i \in \{1, 2\}$, we can generate from L_8 a (G_1, M)-ear square graph, where G_1 and M satisfies i). We first claim that if 1) holds, then the bipartite graph obtained from L_8 by adding the new edges v_0u_2 and v_3u_1 is a (G_1, M)-ear square graph (see Figure 8(b)). The validity of this claims follows from considering $\{v_0u_2, v_3u_1\}$ to be the projected path of $p(x, y)$, $\{v_2u_2, v_3u_3\}$ to be the projected path of $p(w, z)$ and $\{v_0u_0, v_1u_1\}$ to be the projected path of $p(u, v)$.

Secondly, we claim that if 2) holds, then the bipartite graph obtained from L_8 by adding the new edges v_1u_3 and v_0u_2 is a (G_1, M)-ear square graph (see Figure 8(c)). In this case, if we let $\{v_0u_2, v_3u_1\}$ be the projected path of $p(u, v)$, $\{v_2u_2, v_3u_3\}$ be the projected path of $p(w, z)$ and $\{v_0u_0, v_1u_1\}$ be the projected path of $p(x, y)$, then the claim follows.



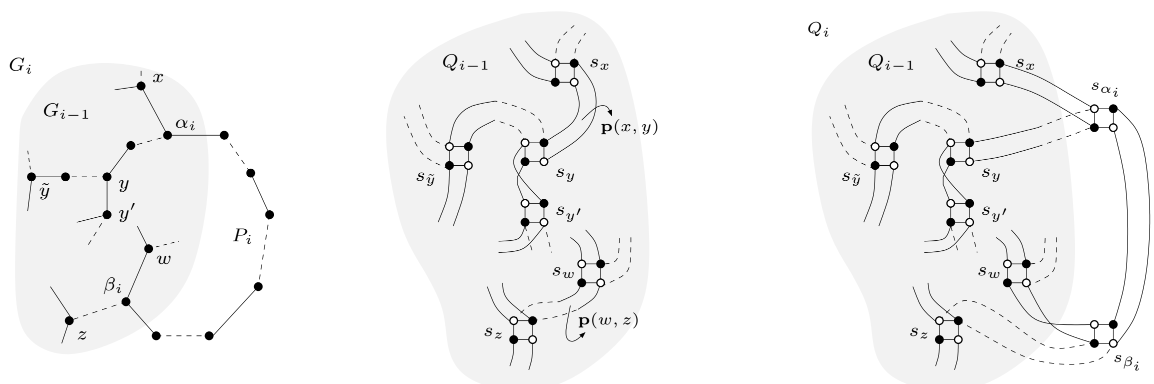
(a) A ladder L_8 on 8 vertices (b) G_1-ear square graph (c) G_1-ear square graph

Figure 8: Generation of Q_1 from L_8.

Proof of Lemma 11, part *2*

For each $i \in \{2, \ldots, l\}$, we need to show that from a (G_{i-1}, M)-ear square graph we can generate a (G_i, M)-ear square graph using simple augmentations. For this purpose, the idea is to make local changes; we basically replace the projected paths in Q_{i-1} of the paths that contain α_i and β_i by two new squares s_{α_i}, s_{β_i}, and by the new projected paths incident with them. Moreover, we modify neither any square in Q_{i-1}, nor the position of the supporting edges of the projected paths incident with them (see Figure 9). Here, α_i, β_i denote the end vertices of the path P_i from $(G_0, G_i, P_i)^l$.

Let $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ be the projected paths in Q_{i-1} such that α_i belongs to $V(p(x, y))$ in G_{i-1} and β_i belongs to $V(p(w, z))$ in G_{i-1}.



(a) $P_i =: \alpha_i \cdots \beta_i$ is a cycle-path of α_i and β_i in G_i. Paths $p(x, y)$ and $p(w, z)$ contain α_i and β_i in G_{i-1}.

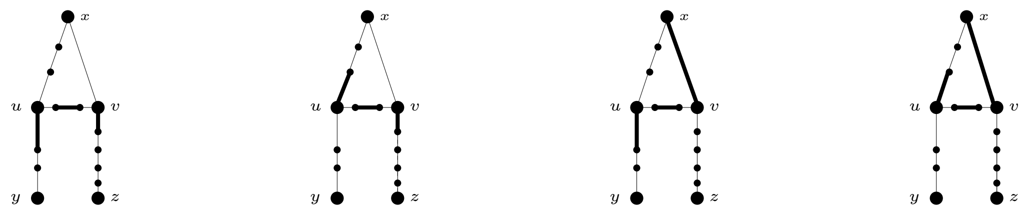
(b) $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ are the projected paths of $p(x, y)$ and $p(w, z)$ in Q_{i-1}.

(c) (G_i, M)-ear square graph. Squares s_{α_i} and s_{β_i} are constructed and also the projected paths incident with them.

Figure 9: $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ are the projected paths in Q_{i-1} such that $\alpha_i \in V(p(x, y))$ and $\beta_i \in V(p(w, z))$ in G_{i-1}. In (a), dashed edges represent the perfect matching M that is absolute in $(G_0, G_i, P_i)^l$. In (b)-(c), dashed lines represent projected matching-paths.

In what follows, for the sake of simplicity we set $u = \alpha_i$ and $v = \beta_i$. We attempt to generate the two new squares s_u and s_v and the projected paths $\mathbf{p}(u, x), \mathbf{p}(u, y), \mathbf{p}(u, v), \mathbf{p}(v, z)$ and $\mathbf{p}(v, w)$. In order to cover all cases we need to take care of two issues, first the interaction between the projected paths $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ in Q_{i-1}, which is described by Proposition 13 and depicted in Figure 7, and the second issue is the location of the perfect matching M with respect to the edges and paths incident with u and v.

For the second issue, we know that P_i is a cycle-path (recall that since M is a perfect matching that is absolute in $(G_0, G_i, P_i)^l$, the path $P_i = p(\alpha_i, \beta_i) \in \mathcal{P}(G_i)$ is always a cycle-path of both u and v in G_i — see Figure 9(a)), and therefore if $p(x, y) \neq p(w, z)$, then the matching-path of u is either $p(u, y)$ or $p(u, x)$ and the matching-path of v is either $p(v, w)$ or $p(v, z)$. In Figure 10, we describe the cases and depict examples for the situation that $x = w$ and $y \neq z$. The remaining cases, for example when all x, w, y, z are different, are analogous.



(a) instance i): $p(u, y)$ matching-path of u and $p(v, z)$ matching-path of v.

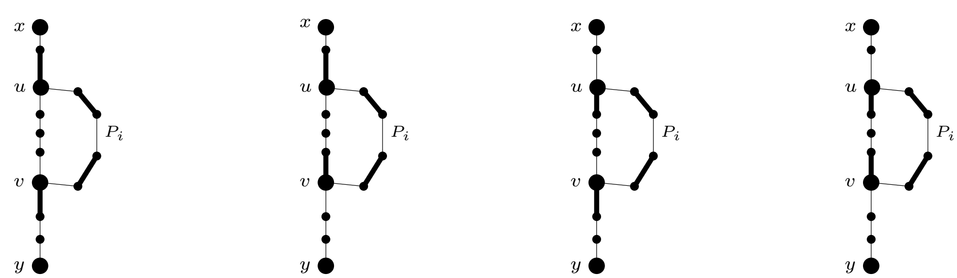
(b) instance ii): $p(u, x)$ matching-path of u and $p(v, z)$ matching-path of v.

(c) instance iii): $p(u, y)$ matching-path of u and $p(v, w)$ matching-path of v.

(d) instance iv): $p(u, x)$ matching-path of u and $p(v, w)$ matching-path of v.

Figure 10: In the subfigures we depicted an example of each instance when the paths $p(x, y)$ and $p(w, z)$ in G_{i-1} intersect in one vertex (see Configurations 2,3 and 4 in Figure 7). Bold edges represent the perfect matching. Moreover, $p(u, v) = P_i$.

In the case that $p(x, y) = p(w, z)$, without loss of generality we can assume that u and v are placed (with respect to x and y) as depicted in Figure 11. Then, we have that the matching-path of u is either $p(u, x)$ or $p(u, v)$ (with $p(u, v) \neq P_i$) and the matching-path of v is either $p(v, y)$ or $p(u, v)$ (with $p(u, v) \neq P_i$). In Figure 11, we describe these situations with a corresponding example.



(a) instance i$'$): $p(u, x)$ matching-path of u and $p(v, y)$ matching-path of v.

(b) instance ii$'$): $p(u, x)$ matching-path of u and $p(u, v)$ matching-path of v.

(c) instance iii$'$): $p(u, y)$ matching-path of u and $p(v, w)$ matching-path of v.

(d) instance iv$'$): $p(u, v)$ matching-path of u and v.

Figure 11: In each figure is depicted an example of the distinct instances in the case that $p(x, y) = p(w, z)$ (see configuration 9 in Figure 7). Bold edges represent the perfect matching.

Summarizing, to prove Lemma 11.*2*, it suffices to prove that from each configuration of the projected paths $\mathbf{p}(x, y)$ and $\mathbf{p}(w, z)$ it is possible to generate all instances i), ii), iii) and iv) in case that $\mathbf{p}(x, y) \neq \mathbf{p}(w, z)$ and that it is possible to generate all instances i$'$), ii$'$), iii$'$) and iv$'$) in case that $\mathbf{p}(x, y) = \mathbf{p}(w, z)$.

Before we go into the analysis of the configurations we shall present an operation consisting of a sequence of simple-augmentations that we constantly use in order to construct two new squares; we shall call this operation a *basic square construction*. This operation is very useful and crucial to reduce the number of cases.

The input of the basic square construction is the bipartite subgraph graph depicted in Figure 12(a) with distinguished edges e_a, e_b, e_c and e_d. The output is the bipartite subgraph depicted in Figure 12(c) with distinguished edges e_a, e_b', e_c' and e_d'. In Figure 12, we depict the sequence of simple-augmentations that compose the basic square construction. It is clear that if H is the graph obtained by applying the basic square construction in a subgraph of a brace, then H is a brace. In what follows, we constantly use this operation and the latter remark.

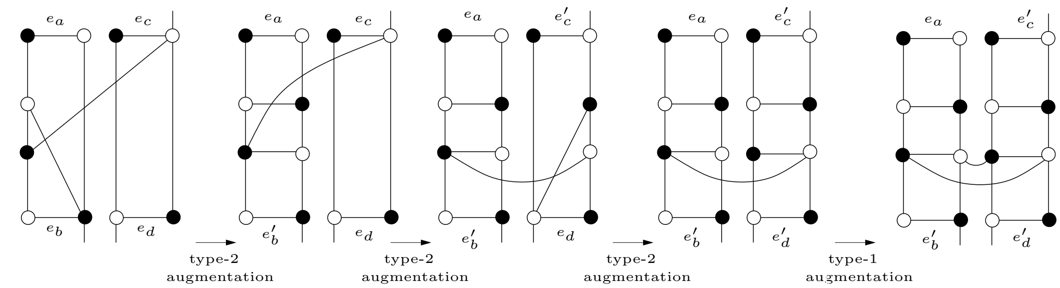


Figure 12: Basic square construction. In subfigure (a) the input subgraph with distinguished edges e_a, e_b, e_c, e_d, in subfigure (b) the steps of the basic square construction and in subfigure (c) the output subgraph with distinguished edges e_a, e_b', e_c', e_d',

Let $V(s_x) = \{x_j : j \in \mathbb{Z}_4\}$, $V(s_y) = \{y_j : j \in \mathbb{Z}_4\}$, $V(s_w) = \{w_j : j \in \mathbb{Z}_4\}$, $V(s_z) = \{z_j : j \in \mathbb{Z}_4\}$. Without loss of generality we assume that $\mathbf{p}(x, y) = \{x_1y_1, x_2y_2\}$, $\mathbf{p}(w, z) = \{w_1z_1, w_2z_2\}$, and that x_1, w_1, y_2, z_2 are in the same partition class, depicted in black in Figure 7. We first study configurations 2, 5, and 7, then 3 and 8, afterwards configurations 4 and 6, and finally configurations 1 and 9.

Configurations 2, 5 and 7 are respectively depicted in Figures 7(b), 7(e) and 7(g). These configurations have a common property, namely: with the notation of Figure 7 each of these configurations may be obtained from Figure 13(a) by possible identifying y_2, z_2 (case of Configuration 5). Next we show how to generate instances i), ii), iii) and iv) of Figure 10.

Generation of instances i) and iv): Let Q^2_{i-1} be the graph obtained from Q_{i-1} by expanding the vertex x_1 to $x_1^1v_1x_1^2$ in such a way that the partition associated with the vertex x_1^2 is either $\{x_2, z_1\}$ if we are generating instance i), or $\{w_2, z_1\}$ if we are generating instance iv). Then, we add the new edge v_1z_2 if we are generating instance i), or v_1y_2 if we are generating instance iv) —see Figures 13(b) and 13(e) without the bold edges for an ilustration of Q^2_{i-1} in each case. Then, we consider the graph $Q^{2,1}_{i-1}$ obtained from Q^2_{i-1} by adding the bold edge. In Figures 13(b) and 13(e), the graph $Q^{2,1}_{i-1}$ is locally depicted for each case. We get the desired instances by applying the basic square construction. We describe this in more details. For instance i): with the notation of Figure 12(a) and 13(b), it is enough to consider $e_a = x_1^1w_2$, $e_b = z_1z_2$, $e_c = x_1^1x_2$ and $e_d = y_1y_2$. For instance iv): with the notation of Figure 12(a) and 13(b), it is enough to consider $e_a = z_1z_2$, $e_b = x_1^1w_2$, $e_c = y_1y_2$ and $e_d = x_1^1x_2$.

For generating instances ii) and iii): Let Q^2_{i-1} be the graph obtained from Q_{i-1} by expanding the vertex x_1 to $x_1^1v_1x_1^2$ in such a way that the partition associated with the vertex x_1^2 is $\{y_1, z_1\}$. Then, we add the new edge v_1z_2 if we are generating instance ii), or v_1y_2 if we are generating instance iii). Then, we consider the graph $Q^{2,1}_{i-1}$ obtained from Q^2_{i-1} by adding the new edge $x_1^1y_1$ if we are generating instance ii) or $x_1^1z_1$ if we are generating instance iii). In Figures 13(c) and 13(d), the graph $Q^{2,1}_{i-1}$ is locally depicted for each case. For instance ii): with the notation of Figure 12(a) and 13(b), it is enough to consider $e_a = x^1_1 w_2$, $e_b = z_1 z_2$, $e_c = y_1 y_2$ and $e_d = x^1_1 x_2$. For instance iii): with the notation of Figure 12(a) and 13(b), it is enough to consider $e_a = x^1_1 x_2$, $e_b = y_1 y_2$, $e_c = x^1_1 w_2$ and $e_d = z_1 z_2$.

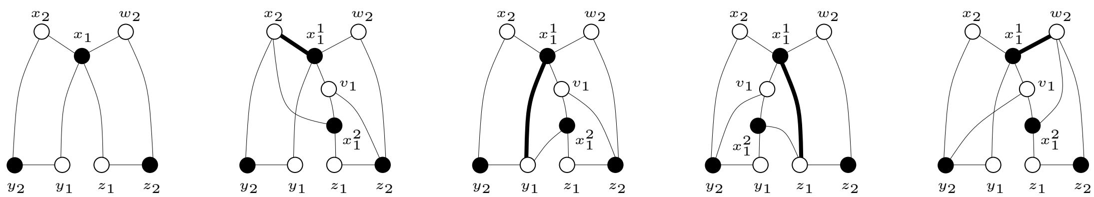


Figure 13: Local view of $Q^{2,1}_{i-1}$ in the generation of instances i), ii), iii) and iv) for configurations 2, 5 and 7. In each configuration, we may possible have $y_2 = z_2$. In (a), for obtaining Configurations 3 and 8 it is enough to identify x_2 and w_2, and delete multiple edges.

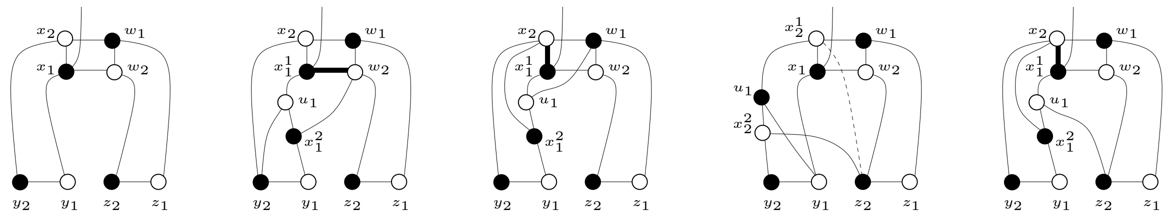
Configurations 3 and 8 are respectively depicted in Figures 7(c) and 7(h). Note that with the notation of Figure 7, both configurations may be obtained from Figure 13(a) by identifying x_2, w_2 and by removing the double edge. Therefore, the reasoning for configurations 2, 5 and 7 applies also for configurations 3 and 8.

Configurations 4 and 6. These configurations are respectively depicted in Figures 7(d) and 7(f). With the notation of Figure 7, both configurations can be locally depicted as in Figure 14(a). Moreover, using the symmetry of both configurations 4 and 6, without loss of generality we can assume that either the degree of x_1 and w_2 in Q_{i-1} is 4 or the degree of x_1 and x_2 in Q_{i-1} is 4. Therefore, in either case we are allowed to expand x_1. Next we show how to generate each instances i), ii), iii) and iv) of Figure 10.

Generation of instance i): Let Q^2_{i-1} be the graph obtained from Q_{i-1} by expanding the vertex x_1 to $x^1_1 u_1 x^2_1$ in such a way that the partition associated with the vertex x^2_1 is $\{w_2, y_1\}$. Then, we add the new edge $u_1 y_2$. Consider the Figure 14(b) without the bold edge for a local ilustration of Q^2_{i-1}. Then, we consider the graph $Q^{2,1}_{i-1}$ obtained from Q^2_{i-1} by adding the new edge $x^1_1 w_2$, namely, the bold edge of Figure 14(b). We finally obtain the desired instance i) by applying the basic square construction in the same fashion as for the case of Configurations 2, 5, and 7.

Generation of instances ii) and iv): Let Q^2_{i-1} be the graph obtained from Q_{i-1} by expanding the vertex x_1 to $x^1_1 u_1 x^2_1$ in such a way that the partition associated with the vertex x^2_1 is $\{x_2, y_1\}$. Then we add the new edge $u_1 w_1$ if we are generating instance ii), or $u_1 z_2$ if we are generating instance iv). Then, we consider the graph $Q^{2,1}_{i-1}$ obtained from Q^2_{i-1} adding the new edge $x^1_1 x_2$ (bold edge in Figures 14(c) and 14(e)). In Figures 14(c) and 14(e), the graph $Q^{2,1}_{i-1}$ for the generation of both instances is locally depicted. Again, we get the desired instances ii) and iv) by applying the basic square construction in the same fashion as for the case of Configurations 2, 5, and 7.

Generation of instance iii): If the edge $x_2 z_2 \notin E(Q_{i-1})$, then add $x_2 z_2$. We denote by Q^1_{i-1} either the graph obtained from Q_{i-1} by adding $x_2 z_2$ or, the graph Q_{i-1} such that $x_2 z_2 \in E(Q_{i-1})$. Hence, z_2, y_2 are neighbors of x_2 in Q^1_{i-1} and clearly $y_2 \neq z_2$ (see Figures 7(d) and 7(f)). Let $Q^{2,1}_{i-1}$ be the graph obtained from Q^1_{i-1} by expanding the vertex x_2 to $x^1_2 u_1 x^2_2$ in such a way that the partition associated with the vertex x^2_2 is $\{z_2, y_2\}$. Then, we add the new edge $u_1 y_1$. Then we obtain $Q^{2,1^*}_{i-1}$ in



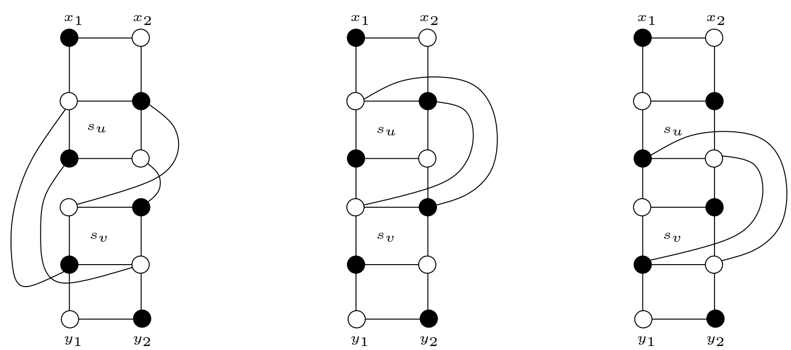
(a) (b) $Q^{2,1}_{i-1}$ for i) (c) $Q^{2,1}_{i-1}$ for ii) (d) $Q^{2,1^*}_{i-1}$ for iii) (e) $Q^{2,1}_{i-1}$ for iv)

Figure 14: Local view of $Q^{2,1}_{i-1}$ or $Q^{2,1^*}_{i-1}$ in the generation of instances i), ii), iii) and iv) for configurations 4 and 6. In each configuration, we have that $y_2 \neq z_2$ and $y_1 \neq z_1$. In case (d), the edge $x_2^1 z_2$ may exist.

the following way: if the edge $x_2z_2 \in E(Q_{i-1})$, then we obtain $Q^{2,1^*}_{i-1}$ from $Q^{2,1}_{i-1}$ by adding the new edge $x_2^1z_2$. Otherwise, $Q^{2,1^*}_{i-1} = Q^{2,1}_{i-1}$. In Figure 14(d) the graph $Q^{2,1^*}_{i-1}$ is locally depicted. Again, we use the basic square construction to complete the generation.

Configuration 1. This configuration is depicted in Figure 7(a). By the symmetry of configuration 1 it suffices to show that we can generate instance iii); this can be generated in the same fashion as the previous instance iii) for configurations 4 and 6.

Configuration 9 is depicted in Figure 7(i). To make things easier, we depict in Figure 15 the subgraphs that we want to generate from Configuration 9; they correspond to the instances i′), ii′), iii′) and iv′) of Figure 11.



(a) Instance i') (b) Instance ii') and iv') (c) Instance iii') and iv')

Figure 15: Instances i'), ii'), iii') and iv') of Figure 11 for configuration 9.

We first focus on the generation of the configurations depicted in Figure 15(b) and Figure 15(c). By symmetry, it suffices to generate only one of them, say we generate the configuration depicted in Figure 15(c).

We split this case into two subcases: (*) at least one vertex of $\{x_1, x_2\}$ has degree 4 in Q_{i-1} and (**) x_1 and x_2 have degree 3 in Q_{i-1}.

subcase (*): without loss of generality we assume that x_1 has degree 4 in Q_{i-1}. Consider the graph $Q^{2,1}_{i-1}$ obtained from Q_{i-1} by expanding x_1 to $x_1^1ux_1^2$ in such a way that the partition associated with the vertex x_1^2 is $\{x_2, y_1\}$. Then we add the new edges uy_2 and $x_2x_1^1$. Next, we consider the graph $Q^{2,1,2}_{i-1}$ obtained from $Q^{2,1}_{i-1}$ by expanding y_2 to $y_2^1vy_2^2$ in such a way that the partition associated with the vertex y_2^2 is $\{u, x_2\}$. Then we add the new edge vx_1^2.

Furthermore, let $Q_{i-1}^{2,1,2,2}$ be the graph obtained from $Q_{i-1}^{2,1,2}$ by expanding x_2 to $x_2^1 u' x_2^2$ in such a way that the partition associated with the vertex x_2^2 is $\{y_2^2, x_1^2\}$. Then we add the new edge uu'. We finally consider the graph $Q_{i-1}^{2,1,2,2,2,1}$ obtained from $Q_{i-1}^{2,1,2,2}$ by expanding u to $u^1 w u^2$ in such a way that the partition associated with the vertex u^2 is $\{u', x_1^1\}$. Then we add the new edges wx_2^2 and wv. The graph $Q_{i-1}^{2,1,2,2,2,1}$ is locally equal to the subgraph depicted in Figure 15(c).

subcase (**): we recall that the set of vertices of the square s_x is given by $\{x_i : i \in \mathbb{Z}_4\}$. Then, by Proposition 12 the vertices x_0 and x_3 have degree 4. Let $Q_{i-1}^{2,1}$ be the graph obtained from Q_{i-1} by expanding x_3 to $x_3^1 u x_3^2$ in such a way that the partition associated with the vertex x_3^2 is $\{x_0, x_2\}$. Then we add the new edges ux_1 and $x_3^1 x_0$. Next, we consider $Q_{i-1}^{2,1,2}$ the graph obtained from $Q_{i-1}^{2,1}$ by expanding x_0 to $x_0^1 vw$ in such a way that the partition associated with the vertex w is $\{x_1, x_3^2\}$. Then we add the new edge vu. Then, we consider $Q_{i-1}^{2,1,2,2}$ the graph obtained from $Q_{i-1}^{2,1,2}$ by expanding u to $u^1 z u^2$ in such a way that the partition associated with the vertex u^2 is $\{x_3^2, x^1\}$. Then we add the new edge zw. We now consider $Q_{i-1}^{2,1,2,2,2,1}$ the graph obtained from $Q_{i-1}^{2,1,2,2}$ by expanding w to $w^1 v' w^2$ in such a way that the partition associated with the vertex w^2 is $\{v, z\}$. Then we add the new edges $u^2 v'$ and $x_2 v'$. The graph $Q_{i-1}^{2,1,2,2,2,1}$ contains the desired instance (see Figure 15(c)).

We now show the generation of the configuration depicted in Figure 15(a). Again we split this case into two subcases: (*) at least one vertex in $\{x_1, x_2, y_1, y_2\}$ has degree 4 in Q_{i-1} and (**) all vertices in $\{x_1, x_2, y_1, y_2\}$ have degree 3 in Q_{i-1}.

subcase (*): without loss of generality we suppose that x_1 has degree 4 in Q_{i-1}. We now consider $Q_{i-1}^{2,1}$ the graph obtained from Q_{i-1} by expanding x_1 to $x_1^1 u x_1^2$ in such a way that the partition associated with the vertex x_1^2 is $\{x_2, y_1\}$. Then we add the new edges uy_2 and $x_1^1 x_2$. Let $Q_{i-1}^{2,1,2}$ be the graph obtained from $Q_{i-1}^{2,1}$ by expanding x_2 to $x_2^1 u' x_2^2$ in such a way that the partition associated with the vertex x_2^2 is $\{x_1^2, y_2\}$. Then we add the new edge uu'. We shall consider $Q_{i-1}^{2,1,2,2}$ obtained from $Q_{i-1}^{2,1,2}$ by expanding y_2 to $y_2^1 v y_2^2$ in such a way that the partition associated with the vertex y_2^2 is $\{u, x_2^2\}$. Then we add the new edge vx_1^2. Let $Q_{i-1}^{2,1,2,2,2,1}$ be the graph obtained from $Q_{i-1}^{2,1,2,2}$ expanding u to $u^1 v' u^2$ in such a way that the partition associated with the vertex u^2 is $\{u', y_2^2\}$. Then we add the new edges $v'v$ and $u^1 u'$. The graph $Q_{i-1}^{2,1,2,2,2,1}$ contains the desired instance (see Figure 15(a)).

subcase (**): by Proposition 12 we have that all vertices in $\{x_0, x_3, y_0, y_3\}$ have degree 4 in Q_{i-1}. We consider Q_{i-1}^2 the graph obtained from Q_{i-1} by expanding x_0 to $x_0^1 u x_0^2$ in such a way that the partition associated with the vertex x_0^2 is $\{x_1, x_3\}$. Then we add the new edges ux_2 and $x_0^1 x_3$. We consider $Q_{i-1}^{2,1,2}$ the graph obtained from $Q_{i-1}^{2,1}$ by expanding x_3 to $x_3^1 u' x_3^2$ in such a way that the partition associated with the vertex x_3^2 is $\{x_0^2, x_2\}$. Then we add the new edge uu'. Let $Q_{i-1}^{2,1,2,2}$ be the graph obtained from $Q_{i-1}^{2,1,2}$ by expanding u to $u^1 v u^2$ in such a way that the partition associated with the vertex u^2 is $\{x_0^2, x_2\}$. Then we add the new edge vx_3^2. We consider $Q_{i-1}^{2,1,2,2,2,1}$ the graph obtained from $Q_{i-1}^{2,1,2,2}$ by expanding x_2 to $x_2^1 v' x_2^2$ in such a way that the partition associated with the vertex x_2^2 is $\{u^2, x_1\}$. Then we add the new edge vv' and $x_1 x_2^1$. The graph $Q_{i-1}^{2,1,2,2,2,1}$ contains the desired instance.

5 Concluding remarks

The aim of this paper is to provide a not-straightforward inductive understanding of the cubic bridgeless graphs. We show how to inductively embed cubic bridgeless graphs into bigger class S of braces obtained from the ladder by simple augmentations. Next step is to generalize theorems and conjectures on cubic bridgeless graphs as statements about elements of S. This is at present our work in progress. In particular, we have used hexagon graphs for study of dcdc. The paper *Cubic bridgeless graphs and braces* is first one in the series of three papers ([2, 3]), where the aim has been to confirm an intuition that "in a sense" all the partial obstacles to validity of dcdc are of cut-type. We managed to confirm this intuition in the last paper ([3]). In this series of papers, we use the machinery of hexagon graphs; not the simple augmentations in a fundamental way though. Nevertheless, the hexagon graphs environment was essential to get into the complex structure of partial obstacles.

As mentioned above, we believe that other conjectures on cubic bridgeless graphs can be formulated for class S, even though at present we do not have a reformulation even for dcdc.

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