Úloha lineárního programování (1)

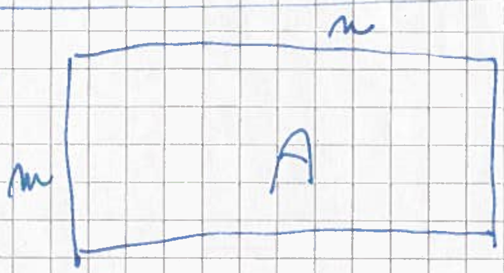
$\max c^T x$
$Ax \leq b$
$x \geq 0$

Standardní tvar (ST)

$\max c^T x$
$Ax = b$
$x \geq 0$

A má lin. nez. řádky

Úmluva:



A reálná matice
b, c reálné vektory

Pozorování. Každou úlohu LP lze převést na ST přidáním pomocných proměnných a vynecháním zbytečných řádků.

Definice. $B \subseteq \{1, 2, \dots, n\}$ je **báze** matice A jestliže $\det A_B \neq 0$.

[úmluva: A_B je podmatice A tvořená sloupci indexovanými prvky B. Stejně pro vektory.]

Definice • $P = \{x ; Ax = b, x \geq 0\}$ je množina přípustných řešení.

• $x \in P$ je **bázické** existuje-li báze B matice A že $x_i = 0$ $\forall i \notin B$. Říkneme, že x patří bázi B [a B je přípustná báze]

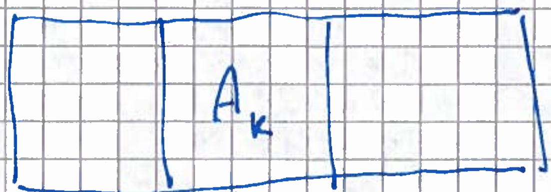
(* on 1A)

• **Pozorování** (jednoduché): Každé bázi patří nejvýše 1 přípustné řešení.

(*) **Pozorování** $x \in P$ je bázické iff sloupce matice A_K jsou lin. nezávislé, kde
$K = \{ j \in \{1,2,\dots,n\} : x_j > 0 \}$.

Důkaz. $\Rightarrow$ z definice

$\Leftarrow$:



$|K| = m \Rightarrow K$ báze

$|K| < m \Rightarrow$ K lze rozšířit na bázi K'

a x je jediné řešení soustavy

$$A_{K'}\, x = b. \quad \boxtimes$$

Důsledek Věty 1. Je-li LP ve ~~tvaru~~ standardním tvaru, má přípustné řešení a omezenou účelovou fci, potom má optimální řešení. [a konečný algoritmus: projdi všechna bázická řešení]

Důsledek. Každý LP s přípustným řešením, který je omezený, má optimální řešení.

~~**Pozorování.** Nechť B je báze. Existuje nejvýše 1 bázické přípustné řešení báze B.~~

Věta 1 Je-li $(c^T x \,;\, Ax = b,\ x \geq 0)$ s aspoň 1 příp. řeš. a shora na množině přípustných řešení omezeno, potom pro každé přípustné řešení x_0 existuje bázické přípustné řešení $\bar{x}$: $c^T x_0 \leq c^T \bar{x}$.

$\boxed{\text{Důkaz: str. 9}}$ Také 1A

Definice. • **Konvexní polyhedron** je průnikem konečně mnoha poloprostorů, t. j. $\{x \in \mathbb{R}^n ;\ Ax \leq b\}$.

$\boxed{\dim P = \max d \text{ t.ž. } x_0, x_1, \dots, x_d \in P \text{ a } x_1 - x_0, x_2 - x_0, \dots, x_d - x_0 \text{ jsou LN}}$

• **Mnohostěn** je omezený konvexní polyhedron. $\boxed{\dim \emptyset \stackrel{\text{def}}{=} -1}$

Definice. Nechť P je konvexní polyhedron.

- $v \in P \subseteq \mathbb{R}^n$ je vrchol jestliže existuje $0 \neq c \in \mathbb{R}^n$ že $c^T v > c^T y$ pro každé $y \in P \setminus \{v\}$.
- $F \subseteq P$ je k-dimenzionální stěna P jestliže F je konvexní polyhedron dimenze k a existuje $0 \neq c \in \mathbb{R}^n$, $r \in \mathbb{R}$, že $(\forall y \in F)(c^T y = r)$ a zároveň $(\forall y \in P \setminus F)(c^T y < r)$.

Věta 2. $P = \{x \in \mathbb{R}^n ;\ Ax = b,\ x \geq 0\}$, $v \in P$.
Je ekvivalentní
- v je vrchol P
- v je bázické přípustné řešení.

$\boxed{\text{Důkaz}}$ str. 9–10

the vertices, and so for such graphs, too, an optimal solution of the LP relaxation tells us almost nothing about the maximum independent set.

It is even known that the size of a maximum independent set cannot be approximated well by any reasonably efficient algorithm whatsoever (provided that some widely believed but unproved assumptions hold, such as $P \neq NP$). This result is from

> J. Håstad: Clique is hard to approximate within $n^{1-\varepsilon}$, *Acta Mathematica* 182(1999) 105–142,

and

> `http://www.nada.kth.se/~viggo/problemlist/compendium.html`

is a comprehensive website for inapproximability results.

4. Theory of Linear Programming: First Steps

4.1 Equational Form

In the introductory chapter we explained how each linear program can be converted to the form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b}.$$

But the simplex method requires a different form, which is usually called the *standard form* in the literature. In this book we introduce a less common, but more descriptive term *equational form*. It looks like this:

Equational form of a linear program:

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0}. \end{array}$$

As usual, $\mathbf{x}$ is the vector of variables, A is a given $m \times n$ matrix, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ are given vectors, and $\mathbf{0}$ is the zero vector, in this case with n components.

The constraints are thus partly equations, and partly inequalities of a very special form $x_j \ge 0$, $j = 1, 2, \ldots, n$, called **nonnegativity constraints**. (Warning: Although we call this form equational, it contains inequalities as well, and these must not be forgotten!)

Let us emphasize that *all* variables in the equational form have to satisfy the nonnegativity constraints.

In problems encountered in practice we often have nonnegativity constraints automatically, since many quantities, such as the amount of consumed cucumber, cannot be negative.

Transformation of an arbitrary linear program to equational form. We illustrate such a transformation for the linear program

$$\begin{array}{ll} \text{maximize} & 3x_1 - 2x_2 \\ \text{subject to} & 2x_1 - x_2 \le 4 \\ & x_1 + 3x_2 \ge 5 \\ & x_2 \ge 0. \end{array}$$

We proceed as follows:

1. In order to convert the inequality $2x_1 - x_2 \leq 4$ to an equation, we introduce a new variable x_3, together with the nonnegativity constraint $x_3 \geq 0$, and we replace the considered inequality by the equation $2x_1 - x_2 + x_3 = 4$. The auxiliary variable x_3, which won't appear anywhere else in the transformed linear program, represents the difference between the right-hand side and the left-hand side of the inequality. Such an auxiliary variable is called a **slack variable**.
2. For the next inequality $x_1 + 3x_2 \geq 5$ we first multiply by -1, which reverses the direction of the inequality. Then we introduce another slack variable x_4 with the nonnegativity constraint $x_4 \geq 0$, and we replace the inequality by the equation $-x_1 - 3x_2 + x_4 = -5$.
3. We are not finished yet: The variable x_1 in the original linear program is allowed to attain both positive and negative values. We introduce two new, nonnegative, variables y_1 and z_1, $y_1 \geq 0$, $z_1 \geq 0$, and we substitute for x_1 the difference $y_1 - z_1$ everywhere. The variable x_1 itself disappears.

The resulting equational form of our linear program is

$$\begin{array}{ll}\text{maximize} & 3y_1 - 3z_1 - 2x_2\\ \text{subject to} & 2y_1 - 2z_1 - x_2 + x_3 = 4\\ & -y_1 + z_1 - 3x_2 + x_4 = -5\\ & y_1 \geq 0,\ z_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.\end{array}$$

So as to comply with the conventions of the equational form in full, we should now rename the variables to $x_1, x_2, \ldots, x_5$.

The presented procedure converts an arbitrary linear program with n variables and m constraints into a linear program in equational form with at most $m + 2n$ variables and m equations (and, of course, nonnegativity constraints for all variables).

Geometry of a linear program in equational form. Let us consider a linear program in equational form:

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

As is derived in linear algebra, the set of all solutions of the system $A\mathbf{x} = \mathbf{b}$ is an affine subspace F of the space $\mathbb{R}^n$. Hence the set of all feasible solutions of the linear program is the intersection of F with the *nonnegative orthant*, which is the set of all points in $\mathbb{R}^n$ with all coordinates nonnegative.[1] The following picture illustrates the geometry of feasible solutions for a linear program with $n = 3$ variables and $m = 1$ equation, namely, the equation $x_1 + x_2 + x_3 = 1$:

orem" of polytope theory). Consequently, every convex polytope can be represented either as the intersection of finitely many half-spaces or as the convex hull of finitely many points.

5. The Simplex Method

In this chapter we explain the simplex method for solving linear programs. We will make use of the terms *equational form* and *basic feasible solution* from the previous chapter.

Gaussian elimination in linear algebra has a fundamental theoretical and didactic significance, as a starting point for further developments. But in practice it has mostly been replaced by more complicated and more efficient algorithms. Similarly, the basic version of the simplex method that we discuss here is not commonly used for solving linear programs in practice. We do not put emphasis on the most efficient possible organization of the computations, but rather we concentrate on the main ideas.

5.1 An Introductory Example

We will first show the simplex method in action on a small concrete example, namely, on the following linear program:

$$\begin{array}{lrcrcl} \text{Maximize} & x_1 & + & x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & & & \leq & 3 \\ & & & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0. \end{array} \tag{5.1}$$

We intentionally do not take a linear program in equational form: The variables are nonnegative, but the inequalities have to be replaced by equations, by introducing slack variables. The equational form is

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & x_1 & + & x_2 & & & & & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\ & x_1 & & & & & + & x_4 & & & = & 3 \\ & & & x_2 & & & & & + & x_5 & = & 2 \\ & & & & & & & & x_1, x_2, \ldots, x_5 & \geq & 0, & \end{array}$$

with the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

In the simplex method we first express each linear program in the form of a *simplex tableau*. In our case we begin with the tableau

$$\begin{array}{rcl} x_3 &=& 1 + x_1 - x_2 \\ x_4 &=& 3 - x_1 \\ x_5 &=& 2 - x_2 \\ \hline z &=& x_1 + x_2 \end{array}$$

The first three rows consist of the equations of the linear program, in which the slack variables have been carried over to the left-hand side and the remaining terms are on the right-hand side. The last row, separated by a line, contains a new variable z, which expresses the objective function.

Each simplex tableau is associated with a certain basic feasible solution. In our case we substitute 0 for the variables x_1 and x_2 from the right-hand side, and without calculation we see that $x_3 = 1, x_4 = 3, x_5 = 2$. This feasible solution is indeed basic with $B = \{3, 4, 5\}$; we note that A_B is the identity matrix. The variables x_3, x_4, x_5 from the left-hand side are basic and the variables x_1, x_2 from the right-hand side are nonbasic. The value of the objective function $z = 0$ corresponding to this basic feasible solution can be read off from the last row of the tableau.

From the initial simplex tableau we will construct a sequence of tableaus of a similar form, by gradually rewriting them according to certain rules. Each tableau will contain the *same* information about the linear program, only written differently. The procedure terminates with a tableau that represents the information so that the desired optimal solution can be read off directly.

Let us go to the first step. We try to increase the value of the objective function by increasing one of the nonbasic variables x_1 or x_2. In the above tableau we observe that increasing the value of x_1 (i.e. making x_1 positive) increases the value of z. The same is true for x_2, because both variables have positive coefficients in the z-row of the tableau. We can choose either x_1 or x_2; let us decide (arbitrarily) for x_2. We will increase it, while x_1 will stay 0.

By how much can we increase x_2? If we want to maintain feasibility, we have to be careful not to let any of the basic variables x_3, x_4, x_5 go below zero. This means that the equations determining x_3, x_4, x_5 may limit the increment of x_2. Let us consider the first equation

$$x_3 = 1 + x_1 - x_2.$$

Together with the implicit constraint $x_3 \geq 0$ it lets us increase x_2 up to the value $x_2 = 1$ (while keeping $x_1 = 0$). The second equation

$$x_4 = 3 - x_1$$

does not limit the increment of x_2 at all, and the third equation

$$x_5 = 2 - x_2$$

allows for an increase of x_2 up to $x_2 = 2$ before x_5 gets negative. The most stringent restriction thus follows from the first equation.

We increase x_2 as much as we can, obtaining $x_2 = 1$ and $x_3 = 0$. From the remaining equations of the tableau we get the values of the other variables:

$$\begin{array}{rcl} x_4 &=& 3 - x_1 = 3 \\ x_5 &=& 2 - x_2 = 1. \end{array}$$

In this new feasible solution x_3 became zero and x_2 nonzero. Quite naturally we thus transfer x_3 to the right-hand side, where the nonbasic variables live, and x_2 to the left-hand side, where the basic variables reside. We do it by means of the most stringent equation $x_3 = 1 + x_1 - x_2$, from which we express

$$x_2 = 1 + x_1 - x_3.$$

We substitute the right-hand side for x_2 into the remaining equations, and we arrive at a new tableau:

$$\begin{array}{rcl} x_2 &=& 1 + x_1 - x_3 \\ x_4 &=& 3 - x_1 \\ x_5 &=& 1 - x_1 + x_3 \\ \hline z &=& 1 + 2x_1 - x_3 \end{array}$$

Here $B = \{2, 4, 5\}$, which corresponds to the basic feasible solution $\mathbf{x} = (0, 1, 0, 3, 1)$ with the value of the objective function $z = 1$.

This process of rewriting one simplex tableau into another is called a **pivot step**. In each pivot step some nonbasic variable, in our case x_2, *enters* the basis, while some basic variable, in our case x_3, *leaves* the basis.

In the new tableau we can further increase the value of the objective function by increasing x_1, while increasing x_3 would lead to a smaller z-value. The first equation does not restrict the increment of x_1 in any way, from the second one we get $x_1 \leq 3$, and from the third one $x_1 \leq 1$, so the strictest limitation is implied by the third equation. Similarly as in the previous step, we express x_1 from it and we substitute this expression into the remaining equations. Thereby x_1 enters the basis and moves to the left-hand side, and x_5 leaves the basis and migrates to the right-hand side. The tableau we obtain is

$$\begin{array}{rcl} x_1 &=& 1 + x_3 - x_5 \\ x_2 &=& 2 - x_5 \\ x_4 &=& 2 - x_3 + x_5 \\ \hline z &=& 3 + x_3 - 2x_5 \end{array}$$

with $B = \{1, 2, 4\}$, basic feasible solution $\mathbf{x} = (1, 2, 0, 2, 0)$, and $z = 3$. After one more pivot step, in which x_3 enters the basis and x_4 leaves it, we arrive at the tableau

$$\begin{array}{rcl} x_1 &=& 3 - x_4 \\ x_2 &=& 2 - x_5 \\ x_3 &=& 2 - x_4 + x_5 \\ \hline z &=& 5 - x_4 - x_5 \end{array}$$

is contained in the set of feasible solutions. It "witnesses" the unboundedness of the linear program, since the objective function attains arbitrarily large values on it. The corresponding semi-infinite ray for the original two-dimensional linear program is drawn thick in the picture above.

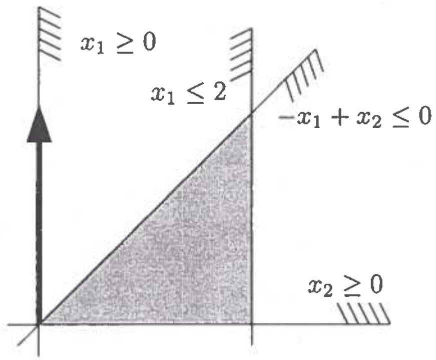
A similar ray is the output of the simplex method for all unbounded linear programs.

5.3 Exception Handling: Degeneracy

While we can make some nonbasic variable arbitrarily large in the unbounded case, the other extreme happens in a situation called a degeneracy: The equations in a tableau do not permit any increment of the selected nonbasic variable, and it may actually be impossible to increase the objective function z in a single pivot step.

Let us consider the linear program

$$\begin{array}{ll} \text{maximize} & x_2 \\ \text{subject to} & -x_1 + x_2 \le 0 \\ & x_1 \le 2 \\ & x_1, x_2 \ge 0. \end{array} \tag{5.2}$$



In the usual way we convert it to equational form and construct the initial tableau

$$\begin{array}{rcrcr} x_3 & = & & x_1 & - \; x_2 \\ x_4 & = & 2 & - \; x_1 & \\ \hline z & = & & & x_2 \end{array}$$

no improvement

The only candidate for entering the basis is x_2, but the first row of the tableau shows that its value cannot be increased without making x_3 negative. Unfortunately, the impossibility of making progress in this case does not imply optimality, so we have to perform a degenerate pivot step, i.e., one with zero progress in the objective function. In our example, bringing x_2 into the basis (with x_3 leaving) results in another tableau with the same basic feasible solution $(0, 0, 0, 2)$:

$$\begin{array}{rcrcr} x_2 & = & & x_1 & - \; x_3 \\ x_4 & = & 2 & - \; x_1 & \\ \hline z & = & & x_1 & - \; x_3 \end{array}$$

Nevertheless, the situation has improved. The nonbasic variable x_1 can now be increased, and by entering it into the basis (replacing x_4) we already obtain the final tableau

$$\begin{array}{rcrcrcr} x_1 & = & 2 & & & - & x_4 \\ x_2 & = & 2 & - & x_3 & - & x_4 \\ \hline z & = & 2 & - & x_3 & - & x_4 \end{array}$$

with an optimal solution $\mathbf{x} = (2, 2, 0, 0)$.

A situation that forces a degenerate pivot step may occur only for a linear program in which several feasible bases correspond to a single basic feasible solution. Such linear programs are called **degenerate**.

It is easily seen that in order that a single basic feasible solution be obtained from several bases, some of the *basic* variables have to be zero.

In this example, after one degenerate pivot step we could again make progress. In general, there might be longer runs of degenerate pivot steps. It may even happen that some tableau is repeated in a sequence of degenerate pivot steps, and so the algorithm might pass through an infinite sequence of tableaus without any progress. This phenomenon is called **cycling**. An example of a linear program for which the simplex method may cycle can be found in Chvátal's textbook cited in Chapter 9 (the smallest possible example has 6 variables and 3 equations), and we will not present it here.

If the simplex method doesn't cycle, then it necessarily finishes in a finite number of steps. This is because there are only finitely many possible simplex tableaus for any given linear program, namely at most $\binom{n}{m}$, which we will prove in Section 5.5.

How can cycling be prevented? This is a nontrivial issue and it will be discussed in Section 5.8.

5.4 Exception Handling: Infeasibility

In order that the simplex method be able to start at all, we need a feasible basis. In examples discussed up until now we got a feasible basis more or less for free. It works this way for all linear programs of the form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0}$$

with $\mathbf{b} \ge \mathbf{0}$. Indeed, the indices of the slack variables introduced in the transformation to equational form can serve as a feasible basis.

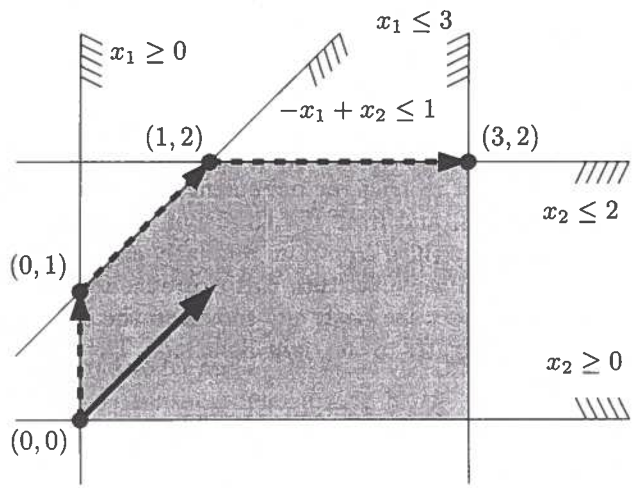
with basis $\{1,2,3\}$, basic feasible solution $\mathbf{x} = (3,2,2,0,0)$, and $z = 5$. In this tableau, no nonbasic variable can be increased without making the objective function value smaller, so we are stuck. Luckily, this also means that we have already found an optimal solution! Why?

Let us consider an *arbitrary* feasible solution $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_5)$ of our linear program, with the objective function attaining some value $\tilde{z}$. Now $\tilde{\mathbf{x}}$ and $\tilde{z}$ satisfy all equations in the final tableau, which was obtained from the original equations of the linear program by equivalent transformations. Hence we necessarily have

$$\tilde{z} = 5 - \tilde{x}_4 - \tilde{x}_5.$$

Together with the nonnegativity constraints $\tilde{x}_4, \tilde{x}_5 \geq 0$ this implies $\tilde{z} \leq 5$. The tableau even delivers a proof that $\mathbf{x} = (3,2,2,0,0)$ is the *only* optimal solution: If $z = 5$, then $x_4 = x_5 = 0$, and this determines the values of the remaining variables uniquely.

A geometric illustration. For each feasible solution (x_1, x_2) of the original linear program (5.1) with inequalities we have exactly one corresponding feasible solution $(x_1, x_2, \ldots, x_5)$ of the modified linear program in equational form, and conversely. The sets of feasible solutions are isomorphic in a suitable sense, and we can thus follow the progress of the simplex method narrated above in a planar picture for the original linear program (5.1):



We can see the simplex method moving along the edges from one feasible solution to another, while the value of the objective function grows until it reaches the optimum. In the example we could also take a shorter route if we decided to increase x_1 instead of x_2 in the first step.

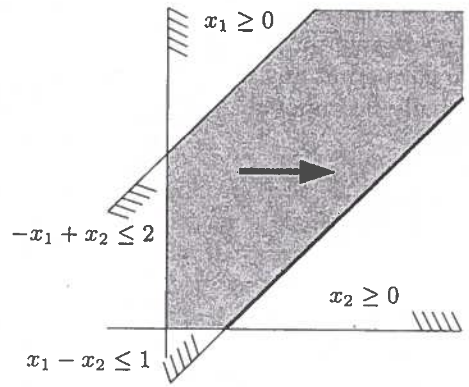
Potential troubles. In our modest example the simplex method has run smoothly without any problems. In general we must deal with several complications. We will demonstrate them on examples in the next sections.

5.2 Exception Handling: Unboundedness

What happens in the simplex method for an unbounded linear program? We will show it on the example

$$\begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

illustrated in the picture below:



After the usual transformation to equational form by introducing slack variables x_3, x_4, we can use these variables as a feasible basis and we obtain the initial simplex tableau

1

$$\begin{array}{rcl} x_3 &=& 1 - x_1 + x_2 \\ x_4 &=& 2 + x_1 - x_2 \\ \hline z &=& \quad\;\; x_1 \end{array}$$

After the first pivot step with entering variable x_1 and leaving variable x_3 the next tableau is

2

$$\begin{array}{rcl} x_1 &=& 1 + x_2 - x_3 \\ x_4 &=& 3 \qquad\;\; - x_3 \\ \hline z &=& 1 + x_2 - x_3 \end{array}$$

If we now try to introduce x_2 into the basis, we discover that none of the equations in the tableau restrict its increase in any way. We can thus take x_2 arbitrarily large, and we also get z arbitrarily large—the linear program is unbounded.

Let us analyze this situation in more detail. From the tableau one can see that for an arbitrarily large number $t \geq 0$ we obtain a feasible solution by setting $x_2 = t$, $x_3 = 0$, $x_1 = 1 + t$, and $x_4 = 3$, with the value of the objective function $z = 1 + t$. In other words, the semi-infinite ray

$$\{(1,0,0,3) + t(1,1,0,0) : t \geq 0\}$$

However, in general, finding any feasible solution of a linear program is equally as difficult as finding an optimal solution (see the remark in Section 1.3). But computing the initial feasible basis can be done by the simplex method itself, if we apply it to a suitable auxiliary problem.

Let us consider the linear program in equational form

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & x_1 + 3x_2 + x_3 = 4 \\ & \phantom{x_1 + {}} 2x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Let us try to produce a feasible solution starting with $(x_1, x_2, x_3) = (0, 0, 0)$. This vector is nonnegative, but of course it is not feasible, since it does not satisfy the equations of the linear program. We introduce auxiliary variables x_4 and x_5 as "corrections" of infeasibility: $x_4 = 4 - x_1 - 3x_2 - x_3$ expresses by how much the original variables x_1, x_2, x_3 fail to satisfy the first equation, and $x_5 = 2 - 2x_2 - x_3$ plays a similar role for the second equation. If we managed to find nonnegative values of x_1, x_2, x_3 for which both of these corrections come out as zeros, we would have a feasible solution of the considered linear program.

The task of finding nonnegative x_1, x_2, x_3 with zero corrections can be captured by a linear program:

$$\begin{array}{llllllll} \text{Maximize} & & & & -x_4 & -x_5 & & \\ \text{subject to} & x_1 & +3x_2 & +x_3 & +x_4 & & = & 4 \\ & & 2x_2 & +x_3 & & +x_5 & = & 2 \\ & x_1, x_2, \ldots, x_5 \geq 0. & & & & & & \end{array}$$

The optimal value of the objective function $-x_4 - x_5$ is 0 exactly if there exist values of x_1, x_2, x_3 with zero corrections, i.e., a feasible solution of the original linear program.

This is the right auxiliary linear program. The variables x_4 and x_5 form a feasible basis, with the basic feasible solution $(0, 0, 0, 4, 2)$. (Here we use that the right-hand sides, 4 and 2, are nonnegative, but since we deal with *equations*, this can always be achieved by sign changes.) Once we express the objective function using the nonbasic variables, that is, in the form $z = -6 + x_1 + 5x_2 + 2x_3$, we can start the simplex method on the auxiliary linear program.

The auxiliary linear program is surely bounded, since the objective function cannot be positive. The simplex method thus computes a basic feasible solution that is optimal.

As training the reader can check that if we let x_1 enter the basis in the first pivot step and x_3 in the second, the final simplex tableau comes out as

$$\begin{array}{rcl} x_1 &=& 2 - x_2 - x_4 + x_5 \\ x_3 &=& 2 - 2x_2 \phantom{{}-x_4} - x_5 \\ \hline z &=& - x_4 - x_5. \end{array}$$

The corresponding optimal solution $(2, 0, 2, 0, 0)$ yields a basic feasible solution of the original linear program: $(x_1, x_2, x_3) = (2, 0, 2)$. The initial simplex tableau for the original linear program can even be obtained from the final tableau of the auxiliary linear program, by leaving out the columns of the auxiliary variables x_4 and x_5,[1] and by changing the objective function back to the original one, expressed in terms of the nonbasic variables:

$$\begin{array}{rcl} x_1 &=& 2 - x_2 \\ x_3 &=& 2 - 2x_2 \\ \hline z &=& 2 + x_2 \end{array}$$

Starting from this tableau, a single pivot step already reaches the optimum.

5.5 Simplex Tableaus in General

In this section and the next one we formulate in general, and mostly with proofs, what has previously been explained on examples.

Let us consider a general linear program in equational form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

The simplex method applied to it computes a sequence of simplex tableaus. Each of them corresponds to a feasible basis B and it determines a basic feasible solution, as we will soon verify. (Let us recall that a feasible basis is an m-element set $B \subseteq \{1, 2, \ldots, n\}$ such that the matrix A_B is nonsingular and the (unique) solution of the system $A_B\mathbf{x}_B = \mathbf{b}$ is nonnegative.)

Formally, we will define a simplex tableau as a certain system of linear equations of a special form, in which the basic variables and the variable z, representing the value of the objective function, stand on the left-hand side and they are expressed in terms of the nonbasic variables.

A **simplex tableau** $\mathcal{T}(B)$ determined by a feasible basis B is a system of $m+1$ linear equations in variables $x_1, x_2, \ldots, x_n$, and z that has the *same set of solutions* as the system $A\mathbf{x} = \mathbf{b}$, $z = \mathbf{c}^T\mathbf{x}$, and in matrix notation looks as follows:

$$\begin{array}{rcl} \mathbf{x}_B &=& \mathbf{p} + Q\mathbf{x}_N \\ \hline z &=& z_0 + \mathbf{r}^T\mathbf{x}_N \end{array}$$

where $\mathbf{x}_B$ is the vector of the basic variables, $N = \{1, 2, \ldots, n\} \setminus B$, $\mathbf{x}_N$ is the vector of nonbasic variables, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{r} \in \mathbb{R}^{n-m}$, Q is an $m \times (n-m)$ matrix, and $z_0 \in \mathbb{R}$.

[1] It may happen that some auxiliary variables are zero but still basic in the final tableau of the auxiliary program, and so they cannot simply be left out. Section 5.6 discusses this (easy) issue.

The basic feasible solution corresponding to this tableau can be read off immediately: It is obtained by substituting $\mathbf{x}_N = \mathbf{0}$; that is, we have $\mathbf{x}_B = \mathbf{p}$. From the feasibility of the basis B we see that $\mathbf{p} \geq \mathbf{0}$. The objective function for this basic feasible solution has value $z_0 + \mathbf{r}^T\mathbf{0} = z_0$.

The values of $\mathbf{p}, Q, \mathbf{r}, z_0$ can easily be expressed using B and $A, \mathbf{b}, \mathbf{c}$:

5.5.1 Lemma. *For each feasible basis B there exists exactly one simplex tableau, and it is given by*

$$Q = -A_B^{-1}A_N,\ \mathbf{p} = A_B^{-1}\mathbf{b},\ z_0 = \mathbf{c}_B^T A_B^{-1}\mathbf{b}, \quad \textit{and} \quad \mathbf{r} = \mathbf{c}_N - (\mathbf{c}_B^T A_B^{-1} A_N)^T.$$

It is neither necessary nor very useful to remember these formulas; they are easily rederived if needed. The proof is not very exciting and we write it more concisely than other parts of the text and we leave some details to a diligent reader. We will proceed similarly with subsequent proofs of a similar kind.

Proof. First let us see how these formulas can be discovered: We rewrite the system $A\mathbf{x} = \mathbf{b}$ to $A_B\mathbf{x}_B = \mathbf{b} - A_N\mathbf{x}_N$, and we multiply it by the inverse matrix A_B^{-1} from the left (these transformations preserve the solution set), which leads to

$$\mathbf{x}_B = A_B^{-1}\mathbf{b} - A_B^{-1}A_N\mathbf{x}_N.$$

We substitute the right-hand side for $\mathbf{x}_B$ into the equation $z = \mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N$, and we obtain

$$\begin{aligned} z &= \mathbf{c}_B^T(A_B^{-1}\mathbf{b} - A_B^{-1}A_N\mathbf{x}_N) + \mathbf{c}_N^T\mathbf{x}_N \\ &= \mathbf{c}_B^T A_B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N)\mathbf{x}_N. \end{aligned}$$

Thus the formulas in the lemma do yield a simplex tableau, and it remains to verify the uniqueness.

Let $\mathbf{p}, Q, \mathbf{r}, z_0$ determine a simplex tableau for a feasible basis B, and let $\mathbf{p}', Q', \mathbf{r}', z_0'$ do as well. Since each choice of $\mathbf{x}_N$ determines $\mathbf{x}_B$ uniquely, the equality $\mathbf{p} + Q\mathbf{x}_N = \mathbf{p}' + Q'\mathbf{x}_N$ has to hold for all $\mathbf{x}_N \in \mathbb{R}^{n-m}$. The choice $\mathbf{x}_N = \mathbf{0}$ gives $\mathbf{p} = \mathbf{p}'$, and if we substitute the unit vectors $\mathbf{e}_j$ of the standard basis for $\mathbf{x}_N$ one by one, we also get $Q = Q'$. The equalities $z_0 = z_0'$ and $\mathbf{r} = \mathbf{r}'$ are proved similarly. $\square$

5.6 The Simplex Method in General

Optimality. Exactly as in the concrete example in Section 5.1, we have the following criterion of optimality of a simplex tableau:

If $\mathcal{T}(B)$ is a simplex tableau such that the coefficients of the nonbasic variables are nonpositive in the last row, i.e., if

$$\mathbf{r} \leq \mathbf{0},$$

then the corresponding basic feasible solution is *optimal.*

Indeed, the basic feasible solution corresponding to such a tableau has the objective function equal to z_0, while for any other feasible solution $\tilde{\mathbf{x}}$ we have $\tilde{\mathbf{x}}_N \geq 0$ and $\mathbf{c}^T\tilde{\mathbf{x}} = z_0 + \mathbf{r}^T\tilde{\mathbf{x}}_N \leq z_0$.

A pivot step: who enters and who leaves. In each step of the simplex method we go from an "old" basis B and simplex tableau $\mathcal{T}(B)$ to a "new" basis B' and the corresponding simplex tableau $\mathcal{T}(B')$. A nonbasic variable x_v enters the basis and a basic variable x_u leaves the basis,[2] and hence $B' = (B \setminus \{u\}) \cup \{v\}$.

We always select the entering variable x_v first.

A nonbasic variable may enter the basis if and only if its coefficient in the last row of the simplex tableau is *positive.*

Only incrementing such nonbasic variables increases the value of the objective function.

Usually there are several positive coefficients in the last row, and hence several possible choices of the entering variable. For the time being the reader may think of this choice as arbitrary. We will discuss ways of selecting one of these possibilities in Section 5.7.

Once we decide that the entering variable is some x_v, it remains to pick the leaving variable.

The leaving variable x_u has to be such that its nonnegativity, together with the corresponding equation in the simplex tableau having x_u on the left-hand side, limits the increment of the entering variable x_v most strictly.

Expressed by a formula, this condition might look complicated because of some double indices, but the idea is simple and we have already seen it in examples. Let us write $B = \{k_1, k_2, \ldots, k_m\}$, $k_1 < k_2 < \cdots < k_m$, and $N = \{\ell_1, \ell_2, \ldots, \ell_{n-m}\}$, $\ell_1 < \ell_2 < \cdots < \ell_{n-m}$. Then the ith equation of the simplex tableau has the form

$$x_{k_i} = p_i + \sum_{j=1}^{n-m} q_{ij} x_{\ell_j}.$$

[2] The letters u and v do not denote vectors here (the alphabet is not that long, after all).

Simplexová Metoda $\left[(\tilde{ST})\ Ax = b,\ z = c^T x\right]$

(1) Nechť $B \subseteq \{1,\dots,n\}$ je báze, ne nutně přípustná.

<u>Tabulka báze B</u> (T_B)

$$\begin{array}{l} x_B = p + Q\,x_N \\ \hline z = z_0 + r^T x_N \end{array}$$

Příslušné řešení úlohy $(\tilde{ST})$: dvojice (x, z) kde $x_N = 0$, $x_B = p$, $z = z_0$

$N = \{1,\dots,n\} \setminus B$

x_B vektor bázických prom.

x_N vektor nebázických prom.

z proměnná

Q $m \times (n-m)$ matice, sloupce indexovány N

p, r vektory

$z_0 \in \mathbb{R}$

(2) Simplexová tabulka vznikne elementárními řádkovými úpravami z $(\tilde{ST})$.

(3) Tudíž množina řešení systému rovnic $(\tilde{ST})$ je stejná jako množina řešení systému rovnic (T_B).

(4) Elementární řádkové úpravy odpovídají násobení maticí zleva. Tudíž:

(5) <u>Pozorování.</u>

$$Q = -A_B^{-1} A_N,\quad p = A_B^{-1} b,\quad z_0 = c_B^T A_B^{-1} b,$$

$$r = c_N - \left(c_B^T A_B^{-1} A_N\right)^T$$

<u>Důkaz.</u> (začátek) Abychom dostali (řád. úpr.) jednotkovou matici místo A_B, musíme A vynásobit zleva A_B^{-1}. Pozorování je důsledek tohoto kroku. $\boxtimes$

úlohy $(\tilde{S}T)$)

! Příslušné řešení báze B (má tvar (x, z).

• x je přípustné pro (ST) právě když $p \geq 0$.

! x je optimální pro (ST) právě když

• $p \geq 0$ a $r \leq 0$.

Důkaz. Množina řešení $(\tilde{S}T)$ je stejná jako množina řešení (T_B).
Nechť $\tilde{x}$ je přípustné řešení (ST), t.j. $A\tilde{x} = b$, $\tilde{x} \geq 0$. Nechť $\tilde{z} = c^T\tilde{x}$.
Potom $(\tilde{x}, \tilde{z})$ je řešení $(\tilde{S}T)$ a tudíž i (T_B). Je-li $r \leq 0$, dostaneme $\tilde{z} \leq z_0$. $\boxtimes$

Přecházení od jedné tabulky k další se děje krokem, kterému se říká PIVOT.
V každém pivotu se báze B upraví takto:

$B' := B \cup \{v\} \setminus \{u\}$, kde

B' značí novou bázi

ⓐ Vstupní proměnná x_v musí splňovat

$r_v > 0$

ⓑ $Q_{uv} < 0$ a $-\frac{p_u}{Q_{uv}} = \min\left\{-\frac{p_i}{Q_{iv}};\ Q_{iv} < 0,\ i = 1, 2, \dots, m\right\}$.

? Co když u neexistuje?

Pozorování Je-li sloupec Q^v matice Q indexovaný $v \in N$ nezáporný, potom je úloha $(\check{S}T)$ neomezená.

Důkaz. Nechť (x, z) je příslušné řešení báze B úlohy $(\check{S}T)$. ~~We let~~, Pro $t \geq 0$ definujme

$$x(t)_B = x_B + Q^v \cdot t, \quad x(t)_v = t, \quad x(t)_w = 0 \text{ pro } w \in N \setminus \{v\}.$$

Potom hodnota cílové funkce pro $x(t)$ je $z_0 + t \cdot r_v$; tato hodnota jde do $+\infty$ pro $t \to +\infty$.
Navíc každé $x(t)$ je přípustné. $\boxtimes$

Předpokládejme tedy že (u, v) existují

! $B' = B \cup \{v\} \setminus \{u\}$ je <u>báze</u> matice A neboť ~~elementár-ními řádkovými~~ ekvivalentními úpravami dostaneme $T_{B'}$ z T_B; a tudíž $(A_{B'})^{-1}$ existuje a tudíž $\det A_{B'} \neq 0$.

! Nechť (x', z') je řešení $(\check{S}T)$ příslušné bázi B'. Potom $x' = x$ nebo $c^T x' > c^T x$.

Důkaz Víme že $p \geq 0$. Jestliže $p_u > 0$, potom $c^T x' > c^T x$. Jestliže $p_u = 0$, dostaneme $x' = x$. $\boxtimes$

Je-li $x' = x$, říkáme že pivot je <u>degenerovaný</u>.

Degenerované pivoty mohou vést k <u>cyklení</u>: po několika degenerovaných krocích můžeme opět dojít k <u>Bázi B</u>. !

Věta 3 Pivot daný Blandovým pravidlem nevede k cyklení.
Důkaz: str. 11

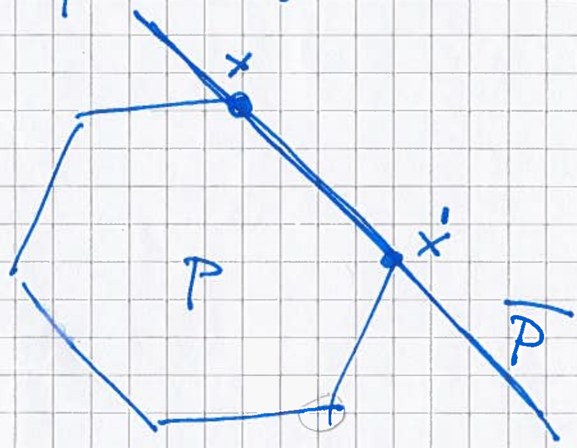
PIVOT geometricky

Nechť $x' \neq x$. $\overline{B} = B \cup \{v\} \Rightarrow |\overline{B}| = m+1$

$\overline{P} = \{ z ; A_{\overline{B}} z_{\overline{B}} = b \}$ je afinní prostor dimenze 1, t. j. přímka, $x \in \overline{P}$, $x' \in \overline{P}$.

Geometrický pohled:

~~$\overline{P}$ definuje stěnu P.~~



$\boxed{\text{Věta}}$ $\overline{P}$ definuje stěnu množiny přípustných řešení.

Důkaz. Definujme $c = (c_1, \dots, c_n)$ předpisem:

$$c_i = 0,\ i \in \overline{B};\quad c_i = -1,\ i \in \{1, \dots, n\} \setminus \overline{B}.$$

Potom $\max(c^T x ; Ax = b, x \geq 0) = 0$, a navíc

- $x \in \overline{P}$, $x \geq 0 \Rightarrow c^T x = 0$.
- $x \notin \overline{P}$, $Ax = b$, $x \geq 0 \Rightarrow \exists i \in \{1, \dots, n\} \setminus \overline{B}$ tak že $x_i > 0$ a tudíž $c^T x < 0$.

Tudíž $\overline{P} \cap \{x ; Ax = b, x \geq 0\}$ je množina všech optimálních řešení, a tudíž je to stěna.

$\boxtimes$

Pivotní Pravidla

(1) Vyber vstupní proměnnou s největším koeficientem (r_v max)

(2) Vyber vstupní proměnnou která vede k největší změně cílové funkce

(3) Nejúspěšnější vyber x' aby

$$\max \frac{c^T(x'-x)}{\|x'-x\|} \quad \left[\|z\| = \sqrt{z_1^2 + \dots + z_n^2} \right]$$

(4) Blandovo pravidlo vždy vyber vstupující v co nejmenší a potom vystupující u co nejmenší.

Věta 3 (připomenutí) Simplexová metoda s Blandovým pravidlem se nezacyklí, t.j., vždy skončí a dá správný výsledek.

[důkaz ve skriptech Dodatky]

Nalezení počáteční přípustné báze

① Je-li LP tvaru $Ax \leq b$, $x \geq 0$, $b \geq 0$ potom počáteční báze je tvořena pomocnými proměnnými které převádí vstupní úlohu na (ST).

② **Obecně**: Máme-li úlohu (ST) $Ax = b$, $x \geq 0$, můžeme předpokládat že $b \geq 0$ [ne $\Rightarrow$ vynásob (-1) příslušné řádky]

- Vyřešíme pomocnou úlohu

$$\max -y_1 - \dots - y_m$$
$$Ax + y = b$$
$$x, y \geq 0 .$$

Počáteční přípustná báze je $\{y_1, \dots, y_m\}$.

Věta (ihned vidět) je ekvivalentní

1. Původní úloha $[\max c^T x \, ; \, Ax = b, \ x \geq 0]$ má přípustné řešení
2. Optimální hodnota cílové fce pomocné proměnné je 0
3. Pomocná úloha má přípustné řešení splňující $y = 0$.

Vyřešme pomocnou úlohu S. Metodou [cílová fce je omezená]. Výsledné bázické řešení, které má hodnotu 0, má $y_1 = \dots = y_m = 0$ a dává přípustné bázické řešení původní úlohy.

Důkazy Vět 1, 2, ~~a~~ 3

Věta 1 Mezi všemi přípustnými řešeními x splňujícími $c^T x \geq c^T x_0$ nechť $\bar{x}$ má nejvíce komponent rovno 0.

Nechť $K = \{ j \leq n ; \bar{x}_j > 0 \}$.

- Jestli jsou sloupce A_K lin. nezávislé, je $\bar{x}$ bázické.
- Jinak nechť $0 \neq v$ splňuje $A_K v = 0$. Rozšíříme v nulami na vektor $w \neq 0$ splňující $Aw = 0$.
- Nejprve nechť w splňuje

$$\boxed{(*) \quad c^T w \geq 0, \quad \exists j \in K,\ w_j < 0.}$$

Definujme $x(t) = \bar{x} + t \cdot w$. Ukážeme že existuje $t_1 > 0$ tak že $x(t_1)$ je přípustné řeš. s více 0-komponentami než $\bar{x}$. (To je spor s výběrem $\bar{x}$ neb) Navíc pro každé t

$$c^T x(t) = c^T \bar{x} + t \cdot c^T w \geq c^T \bar{x}.$$

- Platí $A x(t) = A\bar{x} + t \cdot Aw = b + 0 = b$, pro každé t.
- Je-li $w_j < 0$ ~~pak pro nějaké~~, $j \in K$ pak pro nějaké $t > 0$ bude $x(t)_j = 0$. Vem ~~první t~~ nejmenší $t_1 > 0$ že toto nastane pro nějaké $j \in K$. Pak $x(t_1)$ ~~je~~ ≥ 0, je přípustné a má méně nenulových komponent.

Tím je případ (*) vyřešen.

Nechť (*) nenastane. • $c^T w = 0 \Rightarrow$ w nebo $-w$ splňuje (*).

- Můžeme předp. že $c^T w > 0$ (w nebo $-w$ to splňuje).

(*) neplatí $\Rightarrow w \geq 0$. Pak všechny $x(t)$ jsou přípustná řešení, a $c^T x(t) \xrightarrow[t \to \infty]{} +\infty$. To je spor s omezeností.

$\boxtimes$

$\boxed{\text{Důkaz Věty 2}}$ [v vrchol $\iff$ v bázické řešení] (10)

„$\Rightarrow$" přímo z definice vrcholu a Věty 1

„$\Leftarrow$" v bázické přípustné řešení odpovídající bázi $B \subseteq \{1, \dots, n\}$.

Definujme $\bar{c}$: $\bar{c}_j = 0$, $j \in B$, $\bar{c}_j = -1$, $j \in \{1, \dots, n\} \setminus B$.

Potom $\bar{c}^T v = 0$ a $\bar{c}^T x \leq 0$ pro každé $x \geq 0$. Tudíž v je optimální. Navíc $\bar{c}^T x' < 0$ pro každé $x' \geq 0$ splňující $x'_j > 0$ pro nějaké $j \in \{1, \dots, n\} \setminus B$.

Tudíž jen vektory x splňující $x_{\{1 \dots n\} \setminus B} = 0$ mohou být optimální, a takový přípustný vektor je jediný, protože $A_B x_B = b$ má jediné řešení $x_B = v$ (protože B je báze). $\boxtimes$