$G = (V, E)$ graph (of road network) ①

$\ell : E \to \mathbb{Q}^+$ lengths of edges

Chinese Postman Problem (CPP) [polynomial]

Find a closed route containing all edges at least once, of min total length.

Travelling Salesman Problem (TSP) [NP-complete]

Find a closed route containing all vertices at least once, of min total length.

Ⓐ **CPP is polynomially solvable**

ⓐ G connected, all degrees even $\Rightarrow$ solution to CPP is a <u>closed euler tour</u> which can be found in polynomial time.

ⓑ If G has some vertices of an odd degree then any postman tour traverses some edges multiple times.

$G = (V, E)$ connected, $\ell : E \to \mathbb{Q}^+$

Let $T = \{ v \in V;\ \deg_G(v) \text{ odd} \}$.

* $E' \subseteq E$ is called <u>T-join</u> if graph

$G(T) = (V, E')$ satisfies:

$v \in T \iff \deg_{G(T)}(v)$ odd

In particular, E is T-join.

$\boxed{\text{Observation}}$ Let $E' \subseteq E$ is the ~~edge-set of~~ set of edges of a (min) postman tour which are traversed at least twice.
Then each edge of E' is traversed twice and (V, E') is min T-join for $T = \{ v \in V;\ \deg_G(v) \text{ odd} \}$.

<u>Proof</u>. Observe that E' is set of edges of min total length so that $E \dot\cup E'$ has all degrees even. Hence each edge of E' is traversed twice in the postman tour and Observation follows. $\boxtimes$

<u>Hence</u>: In order to solve CPP, it suffices to find min T-join efficiently.

$\boxed{\text{Algorithm for min } T\text{-join}}$

Make auxiliary graph $H = \left(T, \binom{T}{2}\right)$ and weights $w : \binom{T}{2} \longrightarrow \mathbb{Q}^+$

$w(\{u,v\})$ = length of shortest paths in G from u to v.

$\boxed{\text{Observation}}$ Let P be a perfect matching in H of min total weight. For each $e \in P$ let $p(e)$ be a shortest path in G between the vertices of e. Then $\bigcup_{e \in P} p(e)$ is a min T-join.

<u>Proof</u>. $\bigcup_{e \in P} p(e)$ is clearly a T-join.

If E' is (another) T-join then E' can be partitioned into paths between vertices of T. Hence, there is a perfect matching of H corresponding to E'. This implies $\bigcup_{e \in P} p(e)$ is min T-join.

$\boxtimes$

Travelling Salesman Problem

NP-complete

Additional Assumptions:

A. G is a complete graph

B. $\ell(e) \geq 0$ for each $e \in E$

C. triangular inequality:

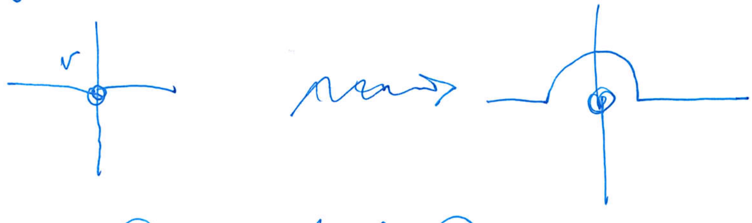
$$(\forall u,v,w \in V)(\ell(u,v) + \ell(v,w) \geq \ell(u,w)).$$

~~Theorem~~ Christofides heuristics

1. let $T \subseteq E$ be min spanning tree
2. $W \subseteq V$ be the set of vertices of odd degree in T
3. let M be a min length perfect matching in $G[W]$ (the graph induced in W).
4. let $J := T \dot\cup M$. Observe J has all <u>degrees even.</u>

a. all degrees are 2 $\Rightarrow$ we have a TS tour

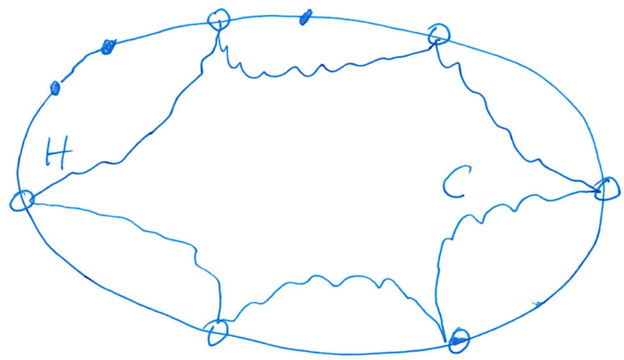
b. otherwise <u>shortcut</u> so that the resulting graph is connected:



c. repeat b until a.

Theorem Assume A, B, C. Then Christofides heuristics gives a TS tour of length at most $3/2$ of shortest TS tour.

<u>Proof</u>. Let H be shortest TS tour, $e \in H$. Then $H \setminus \{e\}$ is a spanning tree and thus $\ell(T) \le \ell(H)$.



o : vertices of W

C: cycle on W following the order of H.

ass.

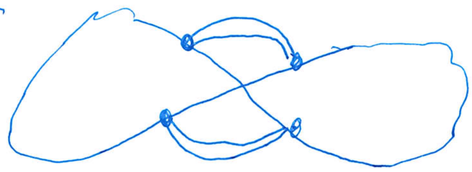
$\ell(C) \le \ell(H)$ by C.

C has an even number of vertices since $|W|$ even and thus C has two perfect matchings. At least one of them has length $\le \frac{\ell(H)}{2}$.

Hence, $\ell(T) + \ell(M) \le 3/2\, \ell(H)$. $\boxtimes$

<u>Other Heuristics.</u>

2-OPT



Lin-Karnighan

Karnighan-Lin

more complicated, very successful.