Exercise 1: Let the scalar product over $\mathbb{C}^3$ be given as:

$\langle \mathbf{x}|\mathbf{y}\rangle = x_1\overline{y_1} + x_2\overline{y_2} + 2x_3\overline{y_3} + x_3\overline{y_2} + x_2\overline{y_3}$

Determine for the following vectors $\mathbf{x}$ and $\mathbf{y}$:

1. the scalar product of $\mathbf{x}$ and $\mathbf{y}$
2. the Euclidean norms of $\mathbf{x}$ and $\mathbf{y}$
3. the distance between $\mathbf{x}$ and $\mathbf{y}$
4. whether vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

a) $\mathbf{x}^T = (4, 2, 3)$, $\mathbf{y}^T = (1, 5, -2)$.

b) $\mathbf{x}^T = (3, 1, -2)$, $\mathbf{y}^T = (1, -3, 2)$.

c) $\mathbf{x}^T = (2, -1, 4)$, $\mathbf{y}^T = (5, 2, -2)$.

d) $\mathbf{x}^T = (2 + i, 0, 4 - 5i)$, $\mathbf{y}^T = (1 + i, 2 + i, -1)$.

Exercise 2: Without calculating the integral show that for any $a, b, r \in \mathbb{R}$, $a, b \neq 0, r > 0$, functions $f_a(x) = \sin(ax)$ and $g_b(x) = \cos(bx)$ are orthogonal.

The product is given as: $\langle f_a|g_b\rangle = \int_{-r}^{r} f_a(x)g_b(x)dx$.

Exercise 3: Let be given two perpendicular vectors $\mathbf{u}$ and $\mathbf{v}$ s.t. $\|\mathbf{u}\| = 12, \|\mathbf{v}\| = 5$. Determine $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u} - \mathbf{v}\|$.

Exercise 4: Prove that $\langle \mathbf{A}|\mathbf{B}\rangle = \mathrm{tr}(\mathbf{B}^T\mathbf{A})$, where tr means the sum of the diagonal entries (so called trace), is a scalar product on real matrices of the same order.

(Matrices $\mathbf{A}, \mathbf{B}$ need not to be square.)

Derive from Cauchy-Schwarz inequality that $\mathrm{tr}(A^2) \leq \mathrm{tr}(A^TA)$.

Exercise 5: Verify, that $\langle \mathbf{x}|\mathbf{y}\rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$ defines a scalar product in $\mathbb{R}^2$.

Exercise 6: Find all unit-length vectors perpendicular to vector $(3, -2)$.

Exercise 7: For a matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix}$$

find a non-zero vector perpendicular

1. to all rows of A,
2. to all vectors in $\mathcal{R}(A)$,
3. to all vectors in $\mathcal{C}(A)$.

Exercise 8: Prove that $xy + yz + zx \leq x^2 + y^2 + z^2$ holds for all $x, y, z \in \mathbb{R}$.