

June 11-15, St Andrews, Scotland

(Conference Preparation Patterns 2007)

Polymerial and quasi-geometric  
Counting

(Charles University, Prague)

Hajto Kla

$f \in S \subseteq A$  (only for  $n^0$ ) (for  $n^1$ ) (for  $n^2$ ) (for  $n^3$ ) (for  $n^4$ ) =  $r_0$

$f(n) = \text{polyomial}(n)$

We can invert such functions

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$S \in f_s, f_s: N \rightarrow N$  counting function, e.g.

$f_s(n) = \text{the } \# \text{ of } A \in S \text{ with size } (A) = n$ .

$f_s = \text{class of enumerative problems}$

---

$$N = \{0, 1, 2, 3, \dots\}$$

- joint work with Lutz Elze

- Some more polynomial classes
- Their connections (simple sets, ...)
- Four (quasi)polynomial classes

### Overview of the talk:



$$\left\lfloor \frac{4}{n^2} \right\rfloor = \ell(n) f \cdot g$$

$a_i : \mathbb{Z} \rightarrow \mathbb{C}$  are periodic functions

e.g.  $f(n) = a_0 + a_1 n + a_2 n^2 + \dots + a_{r-1} n^{r-1}$ , where

$$s.t. n \in \mathbb{Z} \Leftrightarrow f(n) = p(n)$$

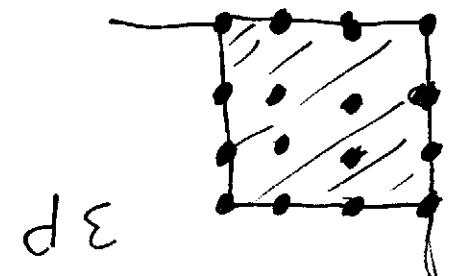
$\Rightarrow$  Is a quasi-polynomial  $p, \dots, p_r$

Four main applications: Storage, Back, Softline, ...

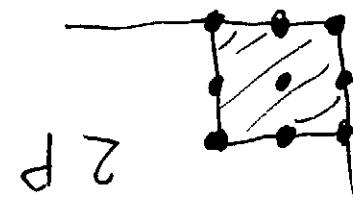
More generally, for rational polynomials  $P$  the function  
? ( $P(n)$ ) is a quasi-periodic function.  
Then (Ehrhart, 62)  $\#(P \cap \mathbb{Z}^d) = \text{poly}(n)$   $\forall n \in \mathbb{N}$ .

Polynomial  $P \in \mathbb{Q}[x]$  where  $nP = \{nx : x \in P\}$ .

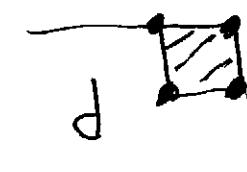
$P \subset \mathbb{R}^d$ ,  $P = \text{conv}(X)$  where finite  $X \subset \mathbb{Z}^d$ ,  $P$  is a lattice.



$3P$



$2P$



$P$

Lattice points in affine polygons

①

(Same for finite sets.)

for  $n < n_0$ . (Same for lower ideals)

$$(n) \subset Q = (n = q_1 + \dots + q_k + 1 = n_1 + q_1 : n_1 \in \mathbb{N}^*) : \# = (n^X : X \in \mathcal{F})$$

in  $(\mathbb{N}^*, \leq)$  then

is  $\mathcal{F}$  is an upper ideal

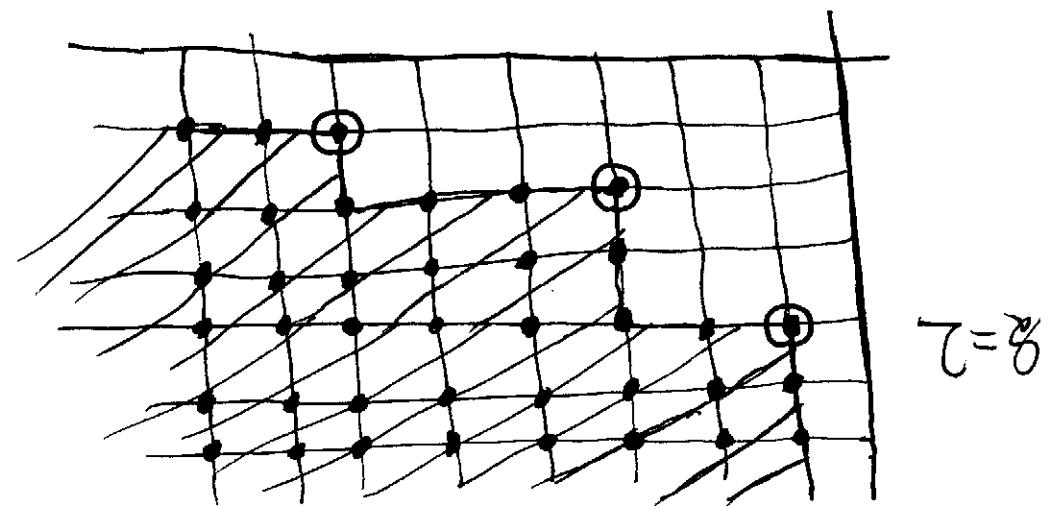
Now  $a = (a_1, \dots, a_n) \leq b = (b_1, \dots, b_n) \Leftrightarrow a_i \leq b_i$

$X \in \mathcal{F} \Leftrightarrow a \in X$

$\mathcal{F}$  is an upper ideal

$(\{ \dots, 1, 0, 1, 0, \dots \} = N) \quad X \subset \mathbb{N}^*$

Lower and upper ideals in a poset



②

☒

$$\text{So } f(n) = \sum_{b \in \mathbb{N}} b^{\#} = \text{poly}(n)$$

polynomialwise

$$k(n) = \max_{a \in \mathbb{N}} (\#(a)) = \text{poly}(n)$$

$$k(n) = \sum_{a \in \mathbb{N}} a^{\#} = O(n) \cdot \sum_{a \in \mathbb{N}} a = O(n)$$

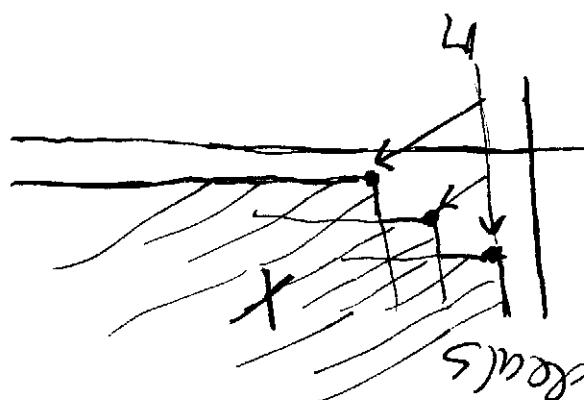
$$(y \cup (n) \Omega)^{\#} = \sum_{n \in N} = (y \cup X)^{\#} = n^{\#} : \exists$$

$$P_n = \{a \in \mathbb{N}^{\mathbb{N}} : \|a\|_n = n\}$$

(Dicksons Lemma, 1912)

$$\ln(\mathbb{N}^{\mathbb{N}})$$

This is an induction  $\rightarrow$  This is finite

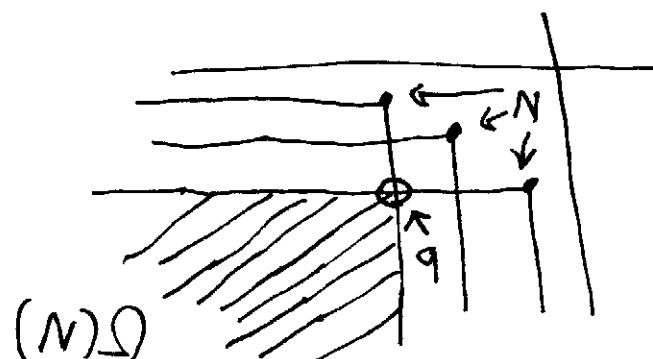


$$M = \min_{a \in \mathbb{N}} (X) \Leftarrow (X) = \bigcap_{a \in \mathbb{N}} Q_a$$

$a \in \mathbb{N}^{\mathbb{N}}$ ,  $Q_a = \{b \in \mathbb{N}^{\mathbb{N}} : b \leq a\} - \text{maximal ideals}$ .  
 $X \subset \mathbb{N}^{\mathbb{N}}$  - an upper ideal.

Proof.

... finite



$$(y \cup n) \Omega^{\#} = (\Omega^{\#} \cup y) \Omega^{\#}$$

$$(y \cup (n) \Omega)^{\#} = \sum_{n \in N} = (y \cup X)^{\#} = n^{\#} : \exists$$

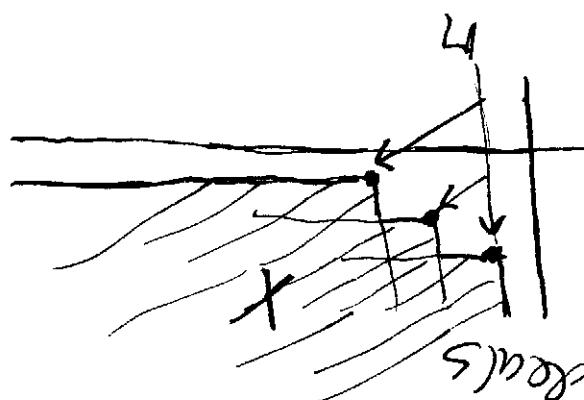
$$k(n) = \sum_{a \in \mathbb{N}} a^{\#} = O(n) \cdot \sum_{a \in \mathbb{N}} a = O(n)$$

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Proof.

3

Sums of sets in semigroups

... + 10

All semigroups considered here are commutative  $\boxed{\bullet = \bullet}$   $\boxed{N+4=N}$ .

$(G, +)$  - (comm.) semigroup, i.e., it is commutative and associative.

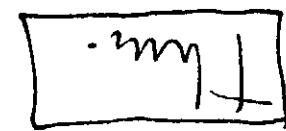
$A, B \subseteq G$ ,  $A+B = \{a+b : a \in A, b \in B\}$

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$\exists n \hookrightarrow \#(n * A) ?$

$\overbrace{A+A+\dots+A}^n = A+A+\dots+A$

The sets in a semigroup, then  $n \hookrightarrow \#(n * A)$  and

 (Khovanskii, 192) If  $A, B \subseteq G$  are finite

$n \hookrightarrow \#(n * A+B)$  are polynomials for  $n < n_0$ .

4

#### ④ Lower ideals of permutations

Let  $y = \bigcup_{n=0}^{\infty} y_n$  be finite permutations so  $y_n = \{a_1, a_2, \dots, a_n\}$ :

( $\mathcal{I}, \subseteq$ ) contains all subsequences order-isomorphic to  $y$ .  
 Permutations:  $y = b_1 a_1 b_2 a_2 \dots b_n a_n \in \mathcal{I}$  if and only if  $b_i \leq a_i$ .

$$\{a_1, a_2, \dots, a_n\} = \{b_1, b_2, \dots, b_n\}$$

Theorem (Kaiserkriterium, 1903)  $\exists f: \mathbb{N} \rightarrow \mathcal{I}$  such that  $f(n) < f(m)$  for  $n < m$ : here  $f_n = f(n)^x$  is the  $n$ -th term of the sequence  $(f_n)_{n \in \mathbb{N}}$ .

numbers,  $(F_n)_{n \geq 0} = (1, 1, 2, 13, 5, 8, 13, 2, \dots)$ ,  $F_n \approx 1.6^n$ .

Examples

- Clear if  $f = f_n$ , because  $|B| \leq |f(B)| \leq |f(f(B))|$ .
- $(G_1, G_2, G_3, \dots)$  is a family of sets such that  $x \in G_i \iff x + a \in G_{i+1}$ . Then  $f_n(B) = n * A + B$ .

Then  $\#(f_n(B)) = p_{G_i}(n)$  for  $n > n_0$ .

and  $f$  be a finite family of countable mappings  $f: X \rightarrow X$ .

Thm. (Kolmanskii, 192) Let  $B \subset X$  be a finite subset

$\{f_i \in f\}$  is a finite family of countable mappings  $f_i: B \rightarrow B$ .

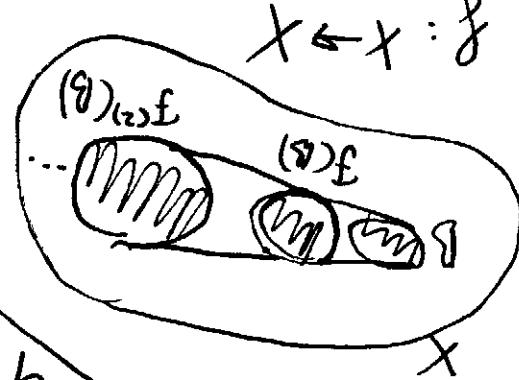
$\cup_{i \in I} (f_1 \circ f_2 \circ \dots \circ f_n)(B) = \{g \in f : g(B) \neq \emptyset\}$

$B \subset X, f: X \rightarrow X, f(B) = \{g \in f : g(B) \neq \emptyset\}$

$(f \circ g) \in f \iff g \circ f = f \circ g$

$X, f \dots$  a family of countably commuting mappings

Kolmanskii's Thm. proved a stronger result.



But what if some  $n_i$ 's are < 0 and some are > 0?

Proof 2: Combinatorial Based on Stirling's Thm. (② above).

First, show  $|n_1 * A_1 + \dots + n_c * A_c| = \text{poly}(n_1, \dots, n_c) A^{n_1} n_1! \dots n_c! A^{n_c}$ .

Thm. (Vatanson, 02) If  $A_1, \dots, A_c, C$  are

for all  $n_1, \dots, n_c \in \mathbb{N}$  Dirichlet then a constant  $C > 0$ .  
Proof 2:  $= \text{poly}(n_1, n_2, \dots, n_c)$  algebraic!

$$|n_1 * A_1 + n_2 * A_2 + \dots + n_c * A_c + B| =$$

First ~~then~~ then

Thm. (Vatanson, 00) If  $A_1, A_2, \dots, A_c, B, C$  are

$(6, +, \dots, (+))$  semi-group

Multivariate generalizations of Klymenko's semi-group theorem.

$\forall a \in A^2$  has  $(c+2)^2$  blocks, which are given. 

(In other words,  $a \neq b \iff a_i b_i < c$ .)

Moreover,  $a_i \leq c \leq b_i$ .

$(a_1 a_2 \dots a_n) \sim (b_1 b_2 \dots b_n)$  iff  $a_i \leq c \leq b_i$

CEA, consider an equivalence relation  $\sim$  on  $A^2$ :

$D_{a,I}$  - equivalence class of  $a$ .

$$D_{a,I} = \{b \in A^2 : b \sim a\}.$$

$$\text{So } D_{a,I} = \{b \in A^2 : b \sim a\} = \{b \in A^2 : a_i \leq b_i \leq a_i\}.$$

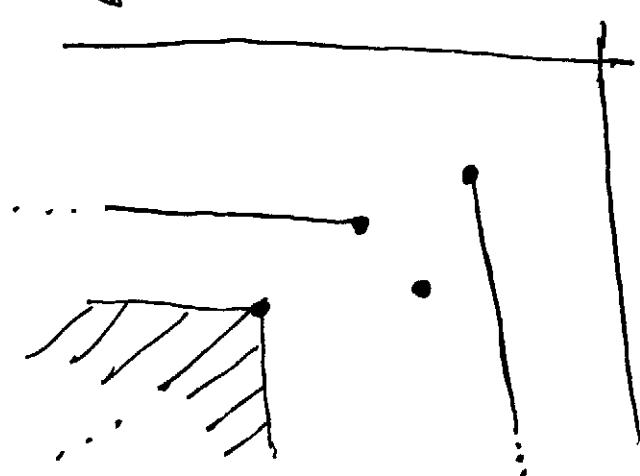
$$\left\{ \begin{array}{l} ?a \leq ?b \leq ?a \\ ?a \sim ?b \end{array} \right\}$$

$$D_{a,I} = \{b \in A^2 : a_i \leq b_i \leq a_i\} = I^a.$$

$$\text{In } A^2 : a \in A^2 \iff I^a = \{b \in A^2 : a_i \leq b_i \leq a_i\}.$$

size of blocks in  $A^2$

$$N^2$$



$(n_1, \dots, n_k) \rightarrow \#(n_1 * A_1 + \dots + n_k * A_k)$  is a SEP function from  $N^k$  to  $N$ .

where  $(G, +)$  is a semigroup, then the function

Thm.  $(\text{DeMorgan's Law}, \text{Def}) \quad f(g(A_1, A_2, \dots, A_k)) = g(A_1, A_2, \dots, A_k)$  are finite,

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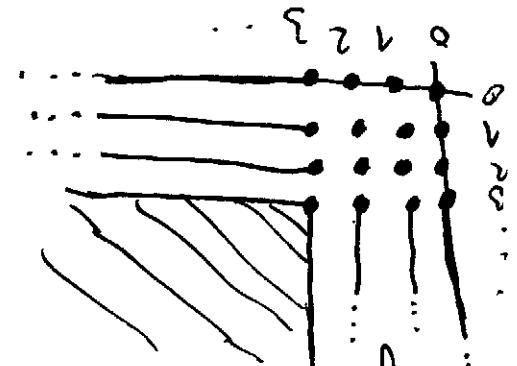
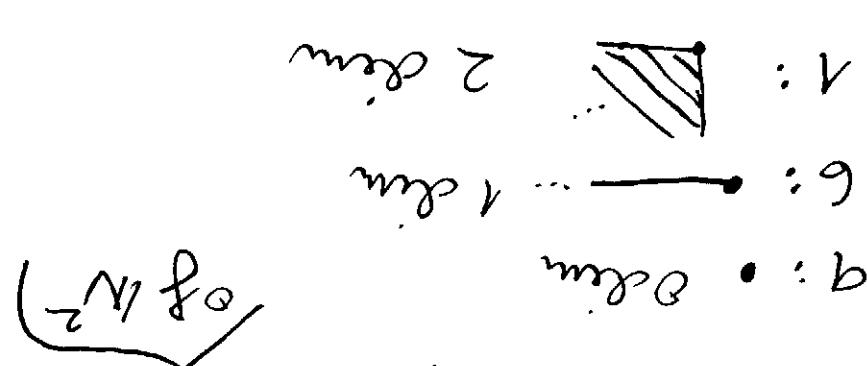
For  $g = 1$  coincides with the usual " $f(u) = \text{Poly}(u)$ " for  $u > c$ :

every gen. element of  $N^{k/n}$

$\exists g \in N$  such that  $g$  is a polynomial function on

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$\text{Def } g: N^k \rightarrow R$  is strongly eventually polynomial (SEP)



12 For example,  $k=2, c=1$  gives this partition into  $Q$ -cells:

We have the following generalization of Steinitz theorem.

$\|a\|^p = a$ , and  $P = \{[a]\}$  gives  $\|a\|^p = \|a\|_1 = (a_1 + a_2 + \dots + a_p)$ .

Moreover  $b_i = \sum a_j$ . For example,  $P = \{[3], [2], \dots, [8]\}$  gives

$$\|a\|^p = (b_1 b_2 \dots b_p) \quad \text{we define}$$

Let  $P = \{P_1, \dots, P_k\}$  be a partition of  $[8]$ . For  $a = (a_1, \dots, a_8) \in U^8$

be unique, finite intersections and complements. It is a good reason

This follows from: The The class of simple sets in  $U^8$  is closed to finite

examples. Finite sets in  $U^8$ , upper ideals (lower ideals), ...

Therefore,  $X = \bigcap_{x \in T} U^{a_x}$  for a finite set  $T$ .

A set  $X \subset U^8$  is simple if it is a finite union of open

Thm. (C.R.K., 167) The following conditions on a set  $X \subset \mathbb{N}^k$  are mutually equivalent.

1.  $X$  is a simple set (= finite union of open intervals).

2. For every partition  $P$  of  $[8]$ , the function

$$(n_1, \dots, n_k) \mapsto \#\{(a \in X : \|a\|_p = (n_1, \dots, n_k)\}$$

3.  $X^* : \mathbb{N}^k \rightarrow \{0, 1\}$  is a SEP function.

4.  $F_x(x_1, \dots, x_k) = \sum_{a \in X} x_1^{a_1} \cdots x_k^{a_k}$  where  $\frac{(1-x_1)(1-x_2) \cdots (1-x_k)}{R(x_1, \dots, x_k)} \in \mathbb{Z}[x_1, \dots, x_k]$ .

Also ideal is a simple set and  $P = \{[8]\}$  gives  $\| \cdot \|_1$ , thus this <sup>upper</sup> ideal shows  $\| \cdot \|_1$ 's Stieltjes theorem in ②.

⊗

and using simple sets.

Proof. Using the same binatorial argument of Nuthanson and Rautenbach

$(n_1, \dots, n_r) \vdash \#(f_{(n_1, \dots, n_r)})$  is a  $\leq^*$  function.

For finite  $B$  and  $f$ ,  
 $\boxed{\text{Thm}}$

size of  $n$ ? mappings from  $P$ .

$f_{(n_1, \dots, n_r)} = \{g_1(f_1(\dots(g_r(b) \dots f_2(\dots(f_1(b)\dots)$ ) :  $b \in B$ ,  $f_i$  is a compo-

a partition of  $f$ ,

$BCX_f = \text{unitaly commuting mappings } f : P \rightarrow P$

On detailed images.

We have also ~~the~~ a multivariate generalization of known results that

$\text{P}_{\text{square}}((f)) \leq \text{poly}(n) \Leftrightarrow \text{Ehrhart's theorem}$ .

Then  $\#(P, X, n) = \#(X(N \cup N')) = \#(\text{colors appearing on the left side points in } N)$  for  $n < n_0$ .

What we can do is let  $X \subseteq N \cup N'$  be additive.

(2. & 3.) Let  $P \in \mathbb{Q}^d$  be a lattice polytope

(1.) If  $X$  defines a congruence in the semigroup  $(N \cup N')$  (equivalently,  $X$  is a semigroup homomorphism)

on the colors  $X(a)$  and  $X(b)$

$X: N \cup N' \rightarrow C$  is an odd filter coloring iff  $X(a+b)$  depends only

on  $a$  or  $b$ . on semi-groups in (3).  $C = (\text{integer})^{S \times \mathbb{Z}}$  of colors.

We have a semi-fibration of Ehrhart's theorem in (1) and Kholodenko's

## Multivariate convolution

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⊗

$$n \cdot P \cap \mathbb{Z}^d = (n - 8) * (\mathbb{Z}^d \cap \mathbb{Z}^d)$$

If  $P \subset \mathbb{R}^d$ ,  $d > 1$ , is a finite polytope and  $n \in \mathbb{Z}$ , then

and a geometric sum:

Proof: Uses Khovalashvili's result  $\#(n * A + B) = \text{poly}(n, d, n_0)$

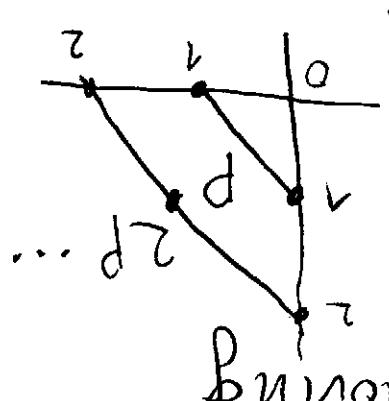
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$$\#(n * A) = \#(\chi(n \cdot P \cap \mathbb{Z}^d))$$

This  $\chi$  and  $P \subset \mathbb{R}^d$  is from ③ because

$$P = \{x \in \mathbb{R}^d : x \geq 0, x_1 + \dots + x_d \leq 1\} - \{x_1 = 0, \dots, x_d = 0\}$$

$$\chi: \mathbb{N}^d \rightarrow \mathbb{G}, (n_1, \dots, n_d) \mapsto n_1 * n_2 * \dots * n_d.$$



$(G, +)$  semigroup,  $A = (a_1, a_2, \dots, a_d) \subset G$  finite, we have collecting

In the last part of my talk, I turn to the thm. on permutations

In (4). Recall that if  $S_n$ :  $\#(\{x \in S_n \mid f(x) = g(x)\}) = \text{poly}(n)$  for all  $n$ ,  
then  $\#(\{x \in S_n \mid f(x) = g(x)\})$  for some  $n$ ;  $X$  is an outer ideal of  
permutations,  $\{f(x) \mid x \in X\} \subseteq \{g(x) \mid x \in X\}$ .

Three proofs:

- in Kacser & Klar, '83
- simpler by Huczynska & Vatter, '06,
- in BalaGya, Boldi & Thoris, '06, as a  
~~special case of a more general and stronger  
result for ordered graphs.~~

Relation to Stanley's thm. in (2):

Thm. Let  $X \subseteq S_n$  be a lower ideal of permutations. If  $f \in F_n$  for  
some  $n$ , then  $E_{f \in X}$  and  $E$  injection  $F: X \rightarrow \{1, \dots, n\}$ .

a)  $F$  is size-preserving ( $\# \rightarrow \text{size}(f) = n$ ),  $F(X)$  is simple set.

$\forall x \in X$  for some  $n < \omega$  ref "  $f_n(x) > f_{n+1}(x)$  " is finite. Thus  $\{f_n(x)\}_{n=0}^{\infty}$  is bounded over  $X$ .

• If  $f$  is uniquely determined by the triple  $(S, \tau, (I_1, I_2, \dots, I_k))$

? If  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_k$  to a single point - words such as uniqueness of  $f$ ?

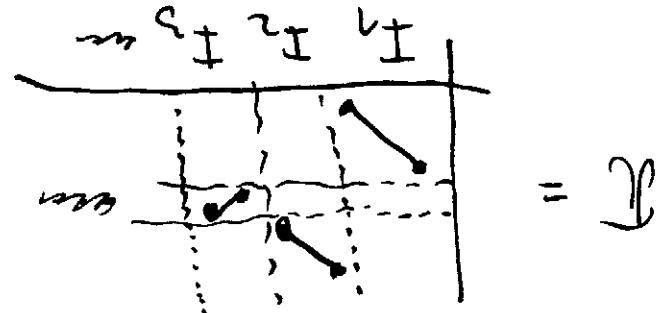
$f(S) = \{f(I_1), f(I_2), \dots, f(I_k)\}$

$f(I_1) = f(I_2) = \dots = f(I_k) = f(S)$

to uniquely determine  $f$  if  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_k$  ... do you understand greedily

$f(I_1) = f(I_2) = \dots = f(I_k) = f(S)$

$f(I_1) = f(I_2) = \dots = f(I_k) = f(S)$



•  $f$  is uniquely determined by  $S, \tau, (I_1, I_2, \dots, I_k)$  if  $I_1 < I_2 < \dots < I_k$  (S.R.B)

Construction of  $E$

Proof:

$\# \text{poly}(n) \text{ for } n < n_0$

$$\#(f) = \#\{g \in X \mid g \sim f\} = \#\{g \in F(x) : \text{len}(g) = n\}$$

Here

Construction of  $E$

That's how

we can do it.



F has the two required properties.

copies

single  $N_k^q$  (large  $k$ )  $\leftarrow \sum \text{copy} (\text{copy})$ .

The copies can be ~~formally~~ <sup>formally</sup> glued to a  
surface

of  $N_k^q$  corresponding to the pair  $(g, z)$ .

F sends  $a$  to the point  $a = (I_1, \dots, I_q)$  in  $\mathbb{R}^q$ .

$\forall z \in X$  we have  $(g, z, (I_1, \dots, I_q))$ .

For every  $(g, z) \in T$  take a copy of  $N_k^q$ . Glue