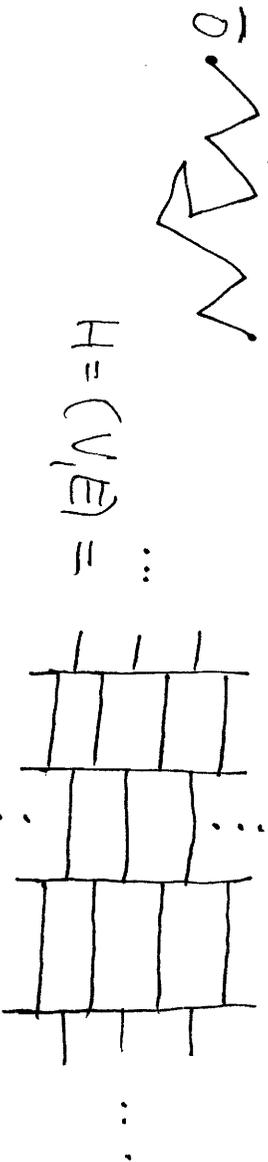


[Thm.] (Duminil-Copin & Smirnov, 2010)

$\Delta_n = \#$   $n$ -step paths in the honeycomb lattice



$$= \mu^{n+\sigma(n)}, \quad \mu = \sqrt{2 + \sqrt{2}} = 2 \cos \frac{\pi}{8} = 1.84775\dots$$

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artiv: [m \(July, 2010\)](#), my rendering also in artiv.

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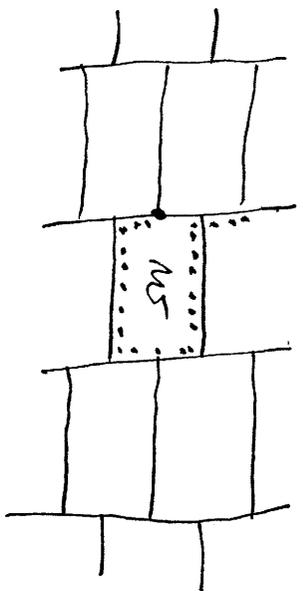
$H$  is transitive  $\rightarrow \Delta_{m+n} \leq \Delta_m \Delta_n \rightarrow \mu = \lim_{n \rightarrow \infty} \frac{\Delta_n}{\mu^n}$  exists and is  $< +\infty$  (Fekete's lemma).

The proof has two parts: 1)  $\mu \geq 2 \cos \frac{\pi}{8}$  and 2)  $\mu \leq 2 \cos \frac{\pi}{8}$ .

Both are based on an identity for the generating function of paths in  $H$ .

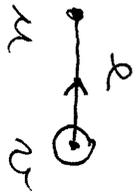
o) The identity

no path in  $H = (V, E)$



lust = 6 (length)

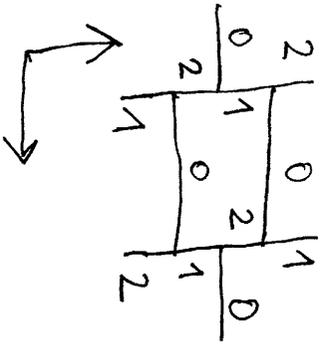
$r(nv) = +3 =$  the winding # of  $nv = \# \downarrow - \# \uparrow$



$\bar{e} = (e, v) =$  oriented edge  
 $\bar{E} = \{ \bar{e} \mid e \in E \}$

$\rho: \bar{E} \rightarrow \{ \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5 \}$

where  $\gamma = e^{2\pi i / 6}$ .  $\rho(\bar{e})$  is the direction of  $\bar{e}$ :

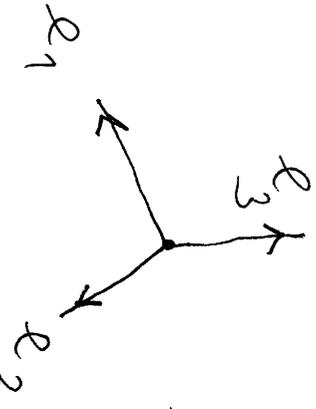


$\rho(\bar{e}) = \gamma^3 \rho(\vec{e}) = -\rho(\vec{e})$ .

Thus direction satisfies:

$\rho(\vec{e}) + \rho(\bar{e}) = 0$  and

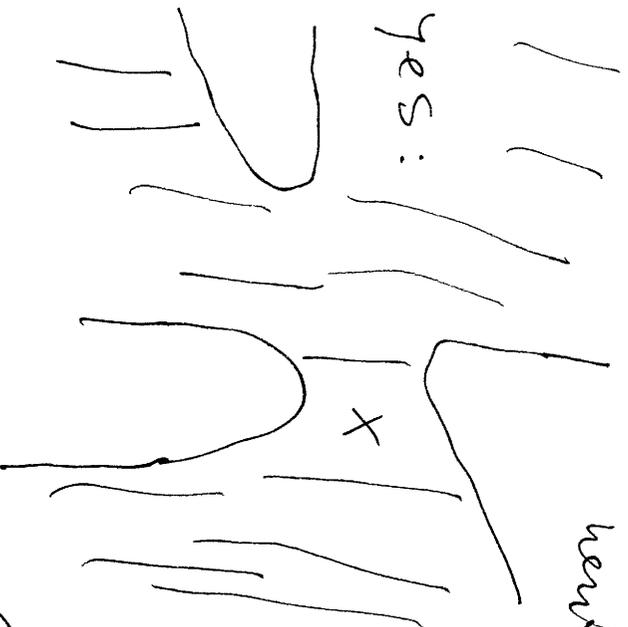
$\rho(\bar{e}_{i+1}) = \gamma^2 \rho(\bar{e}_i)$



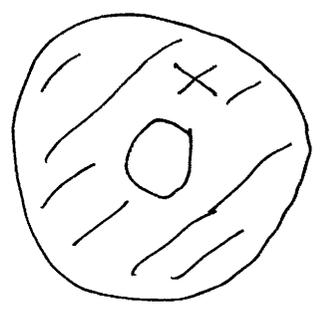
for  $i = 1, 2, 3$  (mod 3)

$X \subset V$  connected, finite or infinite, is a domain

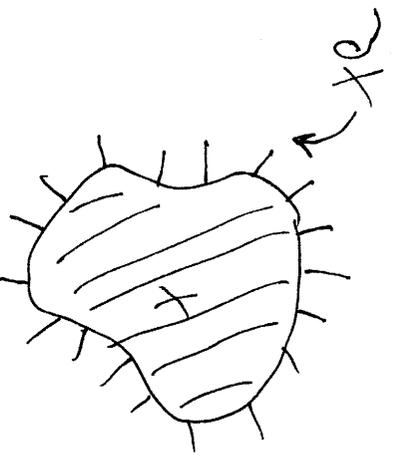
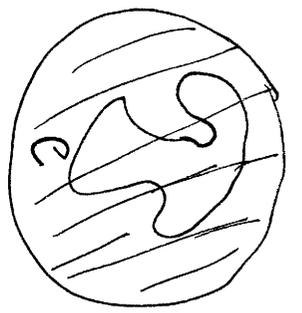
Simple domain:  $H \setminus X$  has only infinite compo-  
nents.



no:



Cycle in  $X$ :



$\partial X =$  the border edges of  $X$

$$a, e \in \partial X, \chi(a, e) := \left\{ \sum_{us} \left| \begin{array}{c} X \\ S_{us} \end{array} \right| \right\}$$

$$F_{a,e}(X) = \sum_{w \in \chi(a,e)} \chi^{(w)} = \sum_{n=1}^{\infty} (\# \text{ paths in } X \text{ from } a \text{ to } e, \text{ length } n) \chi^n$$

E.g.)  $F_{a,a}(X) = X$ . Also, for any  $A \subset \partial X$

$$\text{we set } F_{a,A}(X) = \sum_{e \in A} F_{a,e}(X) = \sum_{w \in \chi(a,A)} \chi^{(w)}$$

We work in  $\mathbb{C}[\gamma, \gamma^{-1}] \llbracket [X] \rrbracket = \left\{ \sum_{h=0}^{\infty} P_h(\gamma) X^h \mid P_h \text{ is a Laurent polynomial} \right\}$ .

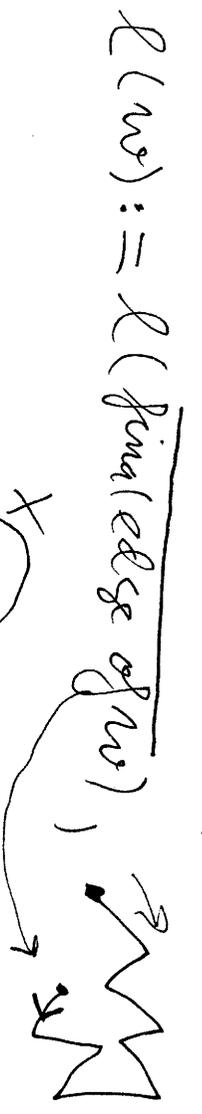
The identity

$H = (V|E)$  the hexagonal graph  
 $X \subset V$  a simple domain  
 $a \in \partial X$  a border edge.

Then

$$\sum_{e \in \partial X} \mathcal{L}(\bar{e}) \gamma^{r(e)} F_{a, e} (X) = (1 + X \xi^2 \gamma + X \xi^2 \gamma^{-1}) \sum_{w \in S} \mathcal{L}(w) \gamma^{|w|} + \sum_{w \in S} \mathcal{L}(w) \gamma^{r(w)} + (\xi^4 \gamma^4 + \xi^4 \gamma^{-4}) \sum_{w \in S} \mathcal{L}(w) \gamma^{|w|} X^{|w|+|C|} \quad C \in \mathcal{C}(w)$$

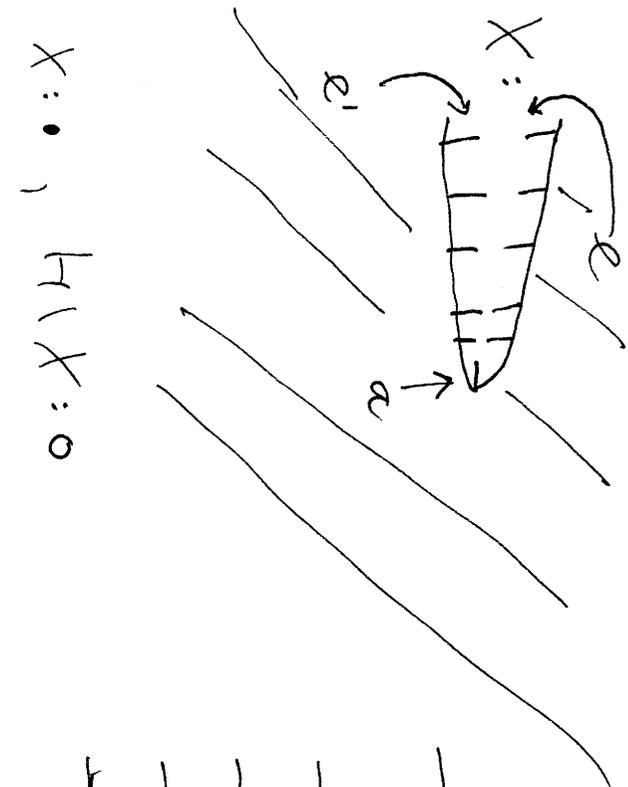
where  $\bar{e} : \begin{matrix} \nearrow \\ \textcircled{X} \end{matrix}$ ,  $r(e) := r(w)$  for any  $w \in X(a, e)$



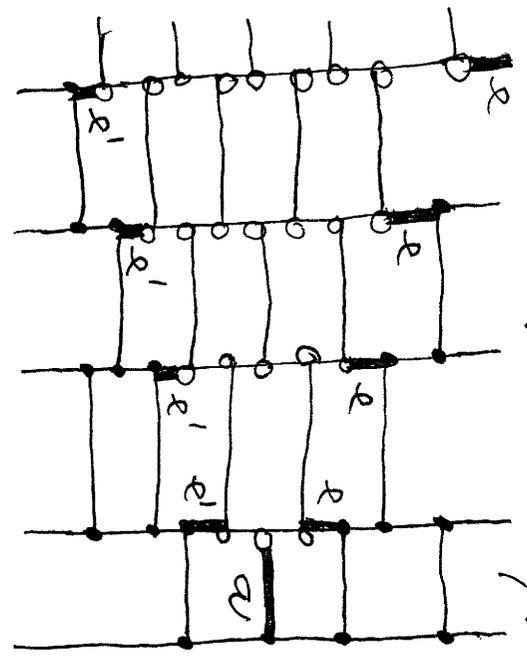
$$S = \{ w \mid \begin{matrix} \nearrow \\ \textcircled{w} \\ \searrow \\ a \end{matrix} \}$$

$$\mathcal{C}(w) = \{ e \mid \begin{matrix} \nearrow \\ \textcircled{w} \\ \searrow \\ e \\ \searrow \\ a \end{matrix} \}$$

1) The lower bound on  $n$



$X: \bullet, H(X) = 0$



More precisely:

$$LS: \mathcal{L}(\bar{e}) y^{r(e)} = -1, \mathcal{L}(\bar{e}) y^{r(e)} = \xi^4 y^4 \text{ and,}$$

by the symmetry  $\Downarrow$ ,  $\mathcal{L}(\bar{e}') y^{r(e')} = (\xi^4 y^4)^{-1}$  and

$F_{a,e}(x) = F(x)$ . Thus

$$\text{The } LS = \sum_{e \in \partial X} \mathcal{L}(\bar{e}) y^{r(e)} F_{a,e}(x) =$$

$$= -X + \underbrace{(\xi^4 y^4 + \xi^{-4} y^{-4})}_{\sum_e F_{a,e}(x)}. \text{ For any } y \in \mathbb{C}$$

such that  $= 0$  the LS equals  $-X$  and on the RS the 2nd term disappears.

$\exists$  quatum  $\xi^4 y^4 + \xi^{-4} y^{-4} = 0$  has 8 solutions;

$$y = \zeta^{1/8}, \quad y_j = \exp(i\pi(j+1)/48), \quad 0 \leq j \leq 7.$$

So for any  $j=0,1,\dots,7$  we have (in  $\mathbb{C}[X]$ )

$$-X = (1 + X \zeta^4 y_j + X \zeta^4 y_j^{-1}) \sum_{w \in S} \ell(w) y_j^{r(w)} X^{|w|}.$$

For  $j=7$ ,  $y_j = e^{2\pi i(43/48)}$  and  $\zeta^4 y_7 + \zeta^{-4} y_7^{-1} = -2 \cos \frac{\pi}{8}$ .  
Hence

$$\sum_{n=1}^{\infty} (2 \cos \frac{\pi}{8})^n X^n = \frac{X}{1 + 2 \operatorname{Re}(\zeta^4 y_7) X} = - \sum_{\substack{5 \leq r \leq 47 \\ r \equiv 0 \pmod{4}}} \sum_{v=0}^r \zeta^v y_7^v G_{\mathbb{Z}_8 \setminus v}(X)$$

where

$$G_{\mathbb{Z}_8 \setminus v}(X) = \sum_{w \in S} X^{|w|}.$$

$$\ell(w) = \zeta^2$$

$$r(w) \equiv v \pmod{48}$$

For  $t_0 = \frac{1}{2 \cos(\pi/8)}$  the geometric series diverges

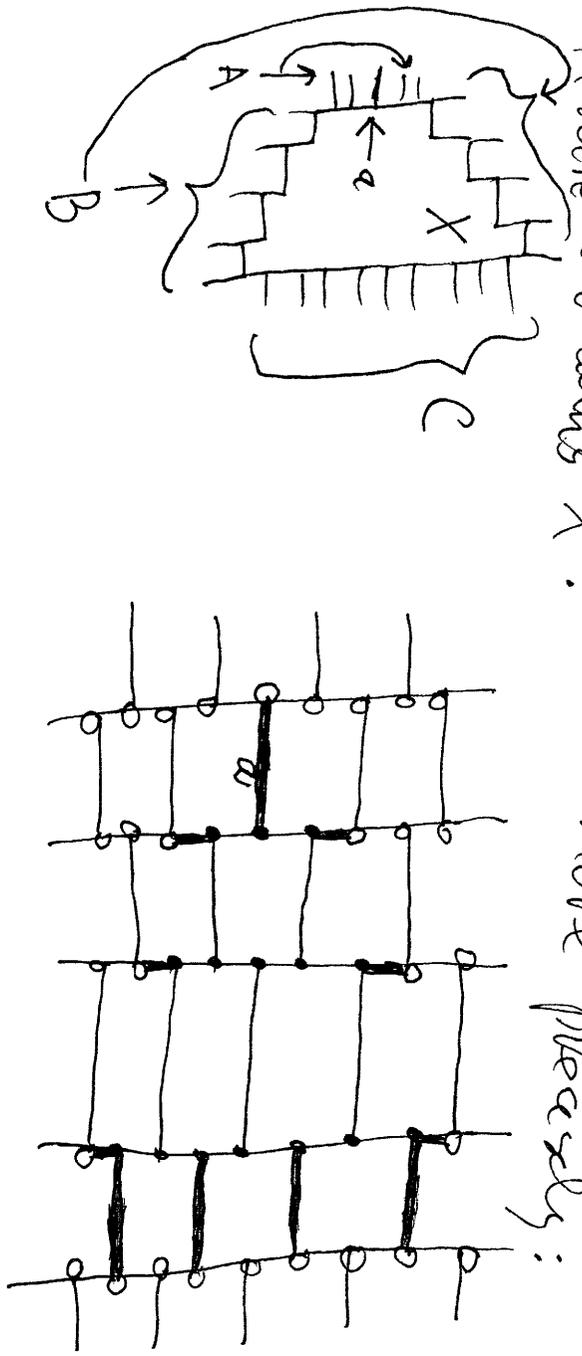
and therefore also  $\log_{\mathbb{Z}_8}(t_0) = +\infty$  for some of

the 288 pairs  $\mathbb{Z}_8, v$ . Since  $\sum_{n=1}^{\infty} \Delta_n X^n \geq G_{\mathbb{Z}_8}(X)$

see otherwise,  $\sum_{n=1}^{\infty} \Delta_n t_0^n = +\infty$  and  $\mu \geq t_0^{-1} = 2 \cos \frac{\pi}{8}$ .

2) The upper bound on  $\mu_c$  (more briefly)

Finite domains  $X$ : More precisely:



$$X: \bullet, H \setminus X: 0.$$

$$LS = \dots = -X + \frac{y^3 + y^{-3}}{2} F_{a,A}(X) + \frac{y^2 + y^{-2} - 2}{2} F_{a,B}(X) + F_{a,C}(X).$$

We set again  $y = y_2$  and now also  $x = x_c = \frac{1}{2 \cos(\frac{\pi}{8})}$  (possible since  $X$  is finite), then  $\mu_c = 0$ .  
Thus

$$\underbrace{\cos\left(\frac{3\pi}{8}\right)}_{>0} F_{a,A}(X_c) + \underbrace{\cos\left(\frac{\pi}{4}\right)}_{>0} F_{a,B}(X_c) + \underbrace{F_{a,C}(X_c)}_{>0} = X_c$$

$$A(50), F_{a,A}(X) = \sum_{n=1}^{\infty} a^n X^n \text{ etc.}$$

It follows that  $V$  ~~is~~ superoid  $X$ ,

$$(\# \text{ use } X(A \cup B \cup C), \text{ vol} = n) \leq \frac{t_c}{\cos(\frac{\pi}{4})} \left(\frac{1}{t_c}\right)^n$$

$\langle X_c^{-n} = (2 \cos \frac{\pi}{8})^n$ ,  $n = 1, 2, 3, \dots$ , No the pre  
uniformity in  $X$ ,

$\leadsto \leadsto$  Extensions to all paths give that  
 $n \leq 2 \cos \frac{\pi}{8}$ .

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