Lecture 6, November 7, 2019

Uniform convergence and normed spaces of functions. Complete spaces of functions. A continuous function without derivative

Normed spaces of functions. In fact, one can consider the pointwise and uniform convergence of sequences of functions for real functions $f\colon M \to \mathbb{R}$ defined on any nonempty set M, which need not be a subset of $\mathbb{R}$ as we stated last time. (For locally uniform convergence one needs to endow M with a metric or a topology, in order to be able to speak about neighborhoods of points.) A better structure to capture uniform convergence than a metric space is in this situation the structure of a *normed (vector) space* $N = (N, \|\cdot\|_\infty) = (N, \|\cdot\|)$ where $N = \{f \mid f\colon M \to \mathbb{R}\}$ and the (supremum or L_∞) norm

$$\|\cdot\|_\infty = \|\cdot\|\colon N \to [0, +\infty] = [0, +\infty) \cup \{+\infty\}$$

is defined by

$$\|f\| := \sup(\{|f(x)| \mid x \in M\}) \ .$$

The norm $\|\cdot\|$ therefore may attain value $+\infty$ and has these basic properties (for every $c \in \mathbb{R}$ and $f, g \in N$): (i) $\|f\| \ge 0$ and $\|f\| = 0$ iff f is the zero function, (ii) $\|cf\| = |c| \cdot \|f\|$, and (iii) $\|f + g\| \le \|f\| + \|g\|$. We understand N as a real vector space with the operations $f + g$ of addition of vectors, which are here functions, and cf of scalar multiplication of a function f by a real number c. The reader knows their properties from the linear algebra. We compute with the not so usual value of the norm $+\infty$ as follows: $c(+\infty) = +\infty$ for every real $c > 0$, $0(+\infty) = 0$, and $(+\infty)+(+\infty) = c + (+\infty) = (+\infty) + c = +\infty$ for every real $c \ge 0$. If we restrict to bounded functions then, as we know,

$$d(f,\, g) := \|f - g\|$$

yields the metric space (N, d).

Uniform convergence and norm. Uniform convergence is equivalent to convergence with respect to the previous norm. The proof of it is left to the reader as a simple exercise.

Proposition ($\rightrightarrows \iff \|\cdot\|_\infty \to 0$). *Let N be the normed space of all real functions defined on a nonempty set M, and let $f_n, f \in N$ for $n \in \mathbb{N}$. Then*

$$f_n \rightrightarrows f \text{ on } M \iff \lim_{n\to\infty} \|f_n - f\| = \lim \|f_n - f\| = \lim d(f_n,\, f) = 0\ ,$$

that is, the functions f_n, as points in the space N, converge in N to the limit f.

Proof. Exercise 1. □

However, here in general we get infinite distances as values of the metric, which we did not allow in the definition of a metric space.

More examples. We saw that for $f_n(x) = x^n$ and f defined as $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$ the sequence of functions $f_n \not\rightrightarrows f$ on $[0,1]$ nor on $[0,1)$. But it is easy to see (Exercise 2) that the previous proposition implies that

$$f_n \overset{\text{loc}}{\rightrightarrows} f \text{ on } [0,1)\ .$$

If $f_n(x) = \frac{nx}{1+n^2x^2}$ then $f_n \to \equiv 0$ on $\mathbb{R}$ but because of the values $f_n(\frac{1}{n}) = \frac{1}{2}$ the convergence is not uniform nor is it on $\mathbb{R}$ locally uniform (Exercise 3). But $f_n \rightrightarrows \equiv 0$ on each set $M \subset \mathbb{R}$ for which 0 is an exterior point (Exercise 4). Finally

$$f_n(x) = \frac{\sin(nx)}{n} \rightrightarrows \equiv 0 \text{ on } \mathbb{R}$$

by the last proposition because $\|f_n\| = \frac{1}{n} \to 0$ ($M = \mathbb{R}$) for $n \to \infty$.

A closed and bounded but non-compact set. We give an example of such a set X in a metric space. As we know, $\mathbb{R}^n$ does not suffice for it and we need infinitely many dimensions. We take the metric space (N, d), where $N = \{f \mid f\colon [0,1] \to \mathbb{R} \text{ and } f \text{ is bounded}\}$ and d is the metric derived from the norm $\|\cdot\|_\infty$, and for $n \in \mathbb{N}$ set

$$f_n(x) = \begin{cases} 0 & \ldots \quad x \ne 1/n\ , \\ 1 & \ldots \quad x = 1/n\ . \end{cases}$$

Then $(f_n) \subset N$ and $d(f_m, f_n) = 1$ whenever $m \ne n$. These distances imply all three required properties of the set $X = \{f_n \mid n \in \mathbb{N}\} \subset N$: it is closed, bounded, and non-compact (Exercise 5).

Proposition (1st complete space of functions). *If M is any non-empty set, then the normed space N of all bounded real functions defined on M is complete, that is, (N, d) with the metric $d(f, g) = \|f - g\|$ is a complete metric space.*

Proof. Let $(f_n) \subset N$ be a Cauchy sequence in the normed space $(N, \|\cdot\|)$, we show that (f_n) has a limit in N. In particular, for each fixed $a \in M$ the sequence $(f_n(a)) \subset \mathbb{R}$ is a Cauchy sequence of real numbers and (by one of the basic theorems in *Mathematical Analysis I*) has a finite limit $\lim f_n(a) =: f(a) \in \mathbb{R}$. We get a function $f\colon M \to \mathbb{R}$ such that $f_n \to f$ on M. We show that even $f_n \rightrightarrows f$ on M.

Let an $\varepsilon > 0$ be given. Because (f_n) is a Cauchy sequence, we can take an n_0 such that for every $m, n \geq n_0$ one has

$$\forall\, x \in M: \ |f_m(x) - f_n(x)| < \varepsilon/2 \ .$$

Further let an $a \in M$ be given. Since $\lim f_n(a) = f(a)$, we can take an m such that $|f_m(a) - f(a)| < \varepsilon/2$ and $m \geq n_0$. Then for every $n \geq n_0$ one has (by the triangle inequality)

$$|f_n(a) - f(a)| \leq |f_n(a) - f_m(a)| + |f_m(a) - f(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \ .$$

The first $|\cdots| < \frac{\varepsilon}{2}$ because the sequence (f_n) is (uniformly) Cauchy, and for the second absolute value it holds too because of the choice of m. For every $a \in M$ and every n with $n \geq n_0$ we therefore have $|f_n(a) - f(a)| < \varepsilon$, thus f_n uniformly converge to f on M, and $\lim f_n = f$ in the normed space of all real functions defined on M.

But we have only bounded functions and therefore still have to show that f is bounded. If it were not, the sequence (f_n) would not have a limit in N. We take an n such that $\|f - f_n\| \leq 1$ (which is possible since $f_n \rightrightarrows f$ on M). Because f_n is bounded, there is a real c with $\|f_n\| \leq c$. By the triangle inequality we have $\|f\| \leq \|f - f_n\| + \|f_n\| \leq 1 + c$, thus f is bounded. □

We required bounded functions in N only in order that an “ordinary” metric results, without infinite distances (Exercise 6).

Theorem (2nd complete space of functions). *For every metric space (M, d) the normed space $N = (N, \|\cdot\|_\infty)$ of all continuous real functions defined on M is complete.*

Proof. Let $(f_n) \subset N$ be a Cauchy sequence in the normed space N of all continuous functions from the metric space M to real numbers, we show that (f_n) has a limit in N. The previous proof shows the existence of a function $f\colon M \to \mathbb{R}$ such that $f_n \rightrightarrows f$ on M. It remains to show that f is continuous.

We show that f is continuous at a given point $a \in M$. Let also an $\varepsilon > 0$ be given. Since $f_n \rightrightarrows f$ on M, we can select an m such that

$$\forall\, x \in M: \ |f_m(x) - f(x)| < \varepsilon/3 \ .$$

Because f_m is a continuous function (at every point of M), there is a $\delta > 0$ such that

$$x \in B(a,\, \delta) \Rightarrow |f_m(x) - f_m(a)| < \varepsilon/3 \ .$$

The ball $B(a, \delta)$ lies in the metric space (M, d). Then for every $x \in B(a, \delta)$ we also have

$$|f(x) - f(a)| \le |f(x) - f_m(x)| + |f_m(x) - f_m(a)| + |f_m(a) - f(a)| < \varepsilon \ .$$

The first and third $|\cdots| < \frac{\varepsilon}{3}$ because of the choice of m, and the second absolute value satisfies it too due to the continuity of f_m at a. Hence f is continuous at a. □

A continuous function on a non-compact metric space (such as $\mathbb{R}$) need not be bounded, hence we get again in general a "metric" $\|f - g\|$ with infinite distances. The theorem actually says that the uniform limit of continuous functions is a continuous function (Exercise 7).

Corollary (the complete space $\mathrm{C}[a, b]$). *For any two real numbers $a \le b$ the normed space $\mathrm{C}[a, b] = (\mathrm{C}[a, b], \|\cdot\|_\infty)$ of all continuous real functions defined on the interval $[a, b]$ is complete.*

Proof. This is a particular case of the previous theorem, but because of the boundedness of continuous real functions on compact sets, and the intervals $[a, b]$ are compact, we now have bounded functions and therefore a "proper" metric with only finite distances. □

The previous complete spaces of functions are derived in a straightforward way from the complete Euclidean space $\mathbb{R}$, the real axis, which makes its role as a basic complete metric space clear.

A continuous function without derivative. Now we give, or at least begin, the promised proof of the existence of a continuous function $f\colon [0, 1] \to$

$\mathbb{R}$ that for no $a \in [0,1]$ has finite $f'(a)$. It applies the Baire theorem in the complete space $\mathrm{C}[0,1]$.

Theorem (a continuous function without derivative). *There exists a continuous function $f\colon [0,1] \to \mathbb{R}$ such that for every $x \in [0,1]$ and every $\delta > 0$,*

$$\sup\left(\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \;\middle|\; y \in P(x,\,\delta) \cap [0,\,1]\right\}\right) = +\infty\ .$$

Such f does not have for any $x \in (0,1)$ finite $f'(x)$, nor it has finite $f'_+(0)$ and finite $f'_-(1)$.

We remind that $P(x,\,\delta) = (x-\delta, x+\delta)\setminus\{x\}$ denotes the deleted neighborhood of the point x. The property $\sup(\dots) = +\infty$ of the function f has the following geometric meaning. For every $x \in [0,1]$, every $\delta > 0$, and every large $c > 0$ (like $c = 10^{100}$), one finds in $[0,1]$ a number y different from x, but closer to x than δ, such that the secant line of the graph of f corresponding to the numbers x and y, which is the line in the plane $\mathbb{R}^2$ going through the points $(x, f(x))$ and $(y, f(y))$, is quite steep, rises or falls with the slope $|\frac{f(y)-f(x)}{y-x}| > c$. Clearly, such function f does not have at the point (number) x finite derivative (a tangent line).

Briefly, the proof goes as follows. For $n \in \mathbb{N}$ we define subsets $A_n \subset \mathrm{C}[0,1]$ by

$$A_n = \{f \in \mathrm{C}[0,1] \mid \exists\, x \in [0,\,1]\; \forall\, y \in [0,\,1]:\; y \neq x \Rightarrow |\tfrac{f(y)-f(x)}{y-x}| \le n\}\ .$$

It turns out that they are meager, and therefore by the Baire theorem there is a function f in $\mathrm{C}[0,1] \setminus \bigcup_{n=1}^{\infty} A_n$. It follows from this that such f has the property in the theorem. I will tell you the proof of the theorem in detail next time.

Exercises

1. Prove the first proposition in the lecture.

2. Prove that $f_n \overset{\mathrm{loc}}{\rightrightarrows} f$ on $[0,1)$, where $f_n(x) = x^n$ and f is the pointwise limit of the functions f_n on $[0,1)$, the zero function.

3. Prove that for $f_n(x) = \frac{nx}{1+n^2x^2}$ the sequence of functions $f_n \overset{\mathrm{loc}}{\not\rightrightarrows} \equiv 0$ on $\mathbb{R}$.

4. Prove that the convergence of the $f_n(x)$ in the previous exercise is uniform on $(-\infty, -\delta) \cup (\delta, +\infty)$ for every $\delta > 0$.

5. Prove that the set of functions $X = \{f_1, f_2, \ldots\} \subset N$ defined in the lecture is closed and bounded but not compact.

6. Prove that the normed space of all real functions defined on a nonempty set is complete.

7. How does exactly follow from the first theorem in the lecture that the uniform limit of continuous functions is a continuous function?

8. Prove that for finite M, $f_n \to f$ on M implies $f_n \rightrightarrows f$ on M.

9. Let $f_n \rightrightarrows f$ on M and $g_n \rightrightarrows g$ on M. Determine if then also $f_n + g_n \rightrightarrows f + g$ on M.

10. Let $f_n \rightrightarrows f$ on M and $g_n \rightrightarrows g$ on M. Determine if then also $f_n g_n \rightrightarrows fg$ on M.