Lecture 1, October 3, 2019

Chapter 1: metric spaces

Definition of metric spaces, isometry. *Metric spaces* formalize phenomenon of distance. A metric space is a pair (M,d) of a set M and a bivariate map

$$d\colon M\times M\to\mathbb{R}$$

called a *metric* or a *distance*, that for any $x,y,z\in M$ satisfies the next three axioms.

a) $d(x,y)=0 \iff x=y$.
b) $d(x,y)=d(y,x)$ (symmetry).
c) $d(x,y)\le d(x,z)+d(z,y)$ (*triangle inequality*).

The axioms easily imply (Exercise 3) that always $d(x,y)\ge 0$ — distances are nonnegative.

An *isometry* of two metric spaces (M,d) and (N,e) is a bijection

$$f\colon M\to N$$

preserving distances: $d(x,y)=e(f(x),f(y))$ for every $x,y\in M$. If such a bijection exists, spaces (M,d) and (N,e) are called *isometric*. It means that they are indistinguishable for all practical purposes.

Examples of metric spaces. Axioms a) and b) are usually easy to verify, but see Exercise 9. To prove the triangle inequality is usually harder, see exercises at the end.

Example 1. $M=\mathbb{R}^n$ for $n\in\mathbb{N}=\{1,2,\dots\}$ and $p\ge 1$ is a real number. We define the distances $d_p(x,y)$ by

$$d_p(x,\,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{1/p}\ ,$$

where $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$. For $n=1$ we get the classical metric $|x-y|$ on $\mathbb{R}$. For $p=2$ and $n\ge 2$ we get the *Euclidean metric*

$$d_2(x,\,y)=\|x-y\|=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}\ .$$

By an *Euclidean space* we mean any metric space of the form (X, d_2), where $X \subset \mathbb{R}^n$, with restricted Euclidean metric. For $p = 1$ and $n \geq 2$ we get the *Manhattan metric*

$$d_1(x,\, y) = \sum_{i=1}^{n} |x_i - y_i| \ ,$$

and for $p \to \infty$ the *maximum metric*

$$d_\infty(x,\, y) = \max_{1 \leq i \leq n} |x_i - y_i| \ .$$

Example 2. M is the set of all bounded functions $f\colon X \to \mathbb{R}$ defined on a set X. Then we have the *supremum metric*

$$d(f,\, g) = \sup_{x \in X} |f(x) - g(x)| \ .$$

If $M = \mathcal{C}[a, b]$ (the set of real functions defined and continuous on the interval $[a, b]$), the supremum is attained and we get the *maximum metric*

$$d(f, g) = \max_{x \in [a,\, b]} |f(x) - g(x)| \ .$$

Example 3. We take $M = \mathcal{C}[a, b]$ and a real number $p \geq 1$. Like in the first example we have metrics

$$d_p(f,\, g) = \left(\int_a^b |f(x) - g(x)|^p \, dx \right)^{1/p} \quad \text{(Riemann integral)} \ .$$

The value $p = 1$ yields the *integral metric* and $p \to \infty$ gives the maximum metric of the second example (Exercise 8). The case $p = 2$ again stands apart. Why is the exponent $p = 2$ exceptional? It turns out that the metric $d_2(\cdot, \cdot)$, both in the 1st and the 3rd example, comes from a scalar product on a vector space and therefore has several nice properties.

For the larger class of functions $M = \mathcal{R}[a, b]$ (Riemann-integrable functions on $[a, b]$) the function $d_p(f, g)$ is well defined, but the axiom a) does not hold and we do not get a metric. Changing the value of a function $f \in \mathcal{R}[a, b]$ in a single point we get a different function $f_0 \in \mathcal{R}[a, b]$, but still $d_p(f, f_0) = 0$. This problem is removed by replacing $\mathcal{R}[a, b]$ with $\mathcal{R}[a, b]/\sim$ for an appropriate equivalence relation $\sim$.

Example 4. A connected graph $G = (M, E)$ with the vertex set M bears a metric

$$d(u, v) = \text{the \# of edges on any shortest path in } G \text{ joining } u \text{ and } v \ .$$

Example 5. For any set (alphabet) A we have on the set $M = A^m$ of all words over A with length m the so called *Hamming metric* or *edit distance* $(u = a_1a_2 \dots a_m, v = b_1b_2 \dots b_m)$

$$d(u, v) = \text{the number of coordinates } i \text{ with } a_i \neq b_i$$

— the minimum number of changes in letters needed to overwrite u in v.

Example 6. On the two-dimensional sphere

$$M = S_2 = \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

we have the metric

$$d(x,\, y) = \text{the length of the shortest curve in } S_2 \text{ joining } x \text{ and } y\ .$$

In more details, for $x \neq y$ the distance $d(x, y)$ equals to the length of the shorter of the two circular arcs of the *main circle* $K(x, y)$ that are determined by the points x and y, and $d(x, x) = 0$. Here $K(x, y)$ is the intersection of S_2 with the plane determined by the origin (which is the center of S_2) and the points x and y on S_2. If these three points are collinear (x and y are antipodes) then $K(x, y)$ is not uniquely determined, but the value $d(x, y) = \pi$ is uniquely determined. Always $0 \leq d(x, y) \leq \pi$. We call this metric the *spherical metric.* One can restrict it to a subset of S_2, for example to the upper hemisphere

$$S_2^+ = \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 \geq 0\}\ .$$

We prove that S_2^+ with the spherical metric is not isometric to any Euclidean space. Thus the same holds for the whole sphere, or more generally for any spherical region containing a hemisphere: it cannot be "squashed" in the plane or any other Euclidean space with all distances preserved.

Proposition (sphere is not flat). *No two metric spaces* (S_2^+, d) *and* (X, d_2)*, where the former is the upper hemisphere with the spherical metric and the latter with* $X \subset \mathbb{R}^n$ *is Euclidean, are isometric.*

Proof. We describe a property of quadruples of points $t, u, v, w \in \mathbb{R}^n$ in the Euclidean space that is not valid in the spherical metric. It is expressed by the implication

$$d_2(t,\, u) = d_2(t,\, v) = d_2(u,\, v) > 0\ \&\ d_2(t,\, w) = d_2(w,\, u) = \frac{d_2(t,\, u)}{2}$$

$$\Rightarrow d_2(w,\, v) = \frac{\sqrt{3} \cdot d_2(t,\, v)}{2} < d_2(t,\, v)\ .$$

Its assumption says that the points t, u, and v form an equilateral triangle with side length $x > 0$, and that w has from both t and u distance $x/2$. For the points t, u, and w the triangle inequality then holds as an equality:

$$d_2(t,\, u) = d_2(t,\, w) + d_2(w,\, u)\ .$$

By the geometry of Euclidean spaces then w lies on the segment tu and at the same time in the hyperplane of points with equal distances to t and u. Therefore w is the midpoint of the segment tu (Exercise 16). Our four points are therefore coplanar (all lie in one plane) and the segment vw is the height in the equilateral triangle from v to the side tu. By the Pythagorean theorem its length $d_2(v, w)$ equals $(\sqrt{3}/2)x$, which is the conclusion of the implication.

If we find on the upper hemisphere four points $t, u, v, w \in S_2^+$ satisfying the assumption of the implication but not the conclusion, we will be done: there is no isometry between the hemisphere and the Euclidean space because every isometry by definition preserves the implication. We take the points

$$t = (1,\, 0,\, 0),\ u = (0,\, 1,\, 0),\ v = (0,\, 0,\, 1),\ \text{and}\ w = (1/\sqrt{2},\, 1/\sqrt{2}, 0)\ .$$

One sees easily that $d(t, u) = d(t, v) = d(u, v) = \frac{\pi}{2}$ and $d(t, w) = d(w, u) = \frac{d(t,u)}{2} = \frac{\pi}{4}$. The point v is the "north pole" ($x_3 = 1$), t, u, and w lie on the "equator" ($x_3 = 0$), and w is the midpoint of the arc tu. But all points on the equator have the same distance $\frac{\pi}{2}$ from the pole v. Hence $d(w, v) = d(t, v)$ and the conclusion of the implication does not hold. $\square$

In nutshell, we have found such four points on the upper hemisphere that their six mutual spherical distances cannot be realized as Euclidean distances. Would not three points be enough (Exercise 12)? And what about replacing the hemisphere with a small spherical region, in which our configuration of four points does not fit (Exercise 11)?

Example 7. Let (M, d) be a metric space. We say that d is a *non-archimedean* metric or an *ultrametric*, if every three points $x, y, z \in M$ satisfy the *strong triangle inequality*

$$d(x,\, y) \leq \max(d(x,\, z),\, d(z,\, y))\ .$$

It is stronger than the triangle inequality, and therefore every ultrametric is a metric. Important properties of any ultrametric are stated in Exercise 14.

The basic example of an ultrametric space is the p-adic metric $(\mathbb{Q}, d_p)$ on the set of fractions (Exercise 13); now p denotes prime numbers, $p = 2, 3, 5, 7, 11, \dots$, and has a different meaning than in Examples 1 and 3. This metric expresses simply in terms of the *p-adic norm*

$$\|\cdot\|_p \colon \mathbb{Q} \to [0, +\infty)$$

as $d_p(x, y) = \|x - y\|_p$. And what is the definition of the p-adic norm? For a nonzero fraction $\frac{a}{b} \in \mathbb{Q}$ as

$$\left\|\frac{a}{b}\right\|_p := \left(\frac{1}{p}\right)^m$$

where $m \in \mathbb{Z}$ is the uniquely determined integer (Exercise 15) such that

$$\frac{a}{b} = p^m \cdot \frac{c}{d},\ \frac{c}{d} \in \mathbb{Q},\ \text{and } p \text{ does not divide } cd\ .$$

For zero we set $\|0\|_p = 0$. Basic properties of the p-adic norm are: (i) $\|\alpha\|_p \ge 0$ and is 0 iff $\alpha = 0$, (ii) $\|\alpha\beta\|_p = \|\alpha\|_p \cdot \|\beta\|_p$, and

$$\text{(iii) } \|\alpha + \beta\|_p \le \max(\|\alpha\|_p,\ \|\beta\|_p)\ .$$

From this one easily deduces that d_p satisfies the strong triangle inequality. The fraction $\frac{1}{p}$ in the definition of $\|\cdot\|_p$ can be replaced by any constant $c \in (0, 1)$ and nothing changes, (i)–(iii) are preserved, but there is a reason for the choice $c = \frac{1}{p}$ which we explain in the next lecture.

Exercises

1. Prove the triangle inequality for the Hamming metric.
2. Prove the triangle inequality for the graph metric.
3. Show that the axioms of a metric space imply that metric is nonnegative.
4. Is symmetry implied by the other axioms of metric?

5. Prove the formula for the maximum metric: $\lim_{p\to+\infty} d_p(x,y) = d_\infty(x,y)$.

6. Deduce from the Cauchy–Schwarz inequality ($a_i, b_i \in \mathbb{R}$)

$$(a_1b_1 + \cdots + a_nb_n)^2 \le (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

the triangle inequality for the Euclidean metric. Prove the Cauchy–Schwarz inequality as well.

7. Check the triangle inequality for the supremum metric.

8. Prove the formula for the maximum metric for functions: if f is continuous on $[a,b]$ then

$$\lim_{p\to+\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} = \max_{x\in[a,b]} |f(x)|\,.$$

9. Verify the axiom a) of metric in Example 3 for $M = \mathcal{C}[a,b]$.

10. Prove the triangle inequality for the spherical metric.

11. Prove that no spherical cap (a part of the sphere S_2 cut off by a plane) with the spherical metric is isometric to an Euclidean space.

12. Can any spherical triangle be isometricly realized in the plane (with Euclidean metric)?

13. Check that the p-adic metric is really an ultrametric.

14. Prove that in an ultrametric space every triangle is isosceles and that if $d(x,z) \ne d(z,y)$ then the strong triangle inequality holds as an equality.

15. Prove that the number $m \in \mathbb{Z}$ from the definition of $\|\cdot\|_p$ is uniquely determined.

16. Prove in detail that if $x,y,z \in \mathbb{R}^n$ satisfy $d_2(x,z) = d_2(z,y) = d_2(x,y)/2 > 0$, then z is the midpoint of the segment xy.