Lecture 13, January 9, 2020

Two Cauchy formulas. Proofs of Theorems 1–3. The residue theorem. Summing $\sum n^{-2k}$

Proof. 1. The linearity of $\int$ follows at once from the linearity of $\int_{\partial R}$. 2. We take some rectangles R_n containing a inside and shrinking to it. The ML bounds show that the integrals $\int_{\partial R_n} f$ go to 0 because $\mathrm{per}(R_n) \to 0$ as $n \to \infty$ and $|f|$ is bounded on the deleted neighborhood. So $\int f = 0$. 3. It is easy to see that if S is the square with vertices $\pm 1 \pm i$ and $a + S$ is its shift, then by the definition of $\int_{\partial R}$ we have $\int_{\partial(a+S)} \frac{1}{z-a} = \int_{\partial S} \frac{1}{z} = \rho$. 4. The ML bounds on the integrals $\int_{\partial R}(f - f_n) = \int_{\partial R} f - \int_{\partial R} f_n$ show that they go to 0: $\mathrm{per}(R)$ is constant but now $\max_{z \in \partial R} |f(z) - f_n(z)| \to 0$ as $n \to \infty$. $\square$

We will need a supplement to the proposition on the constant ρ.

Proposition (other negative powers). *Let R be a rectangle, $a \in \mathrm{int}(R)$ be a point, and $k \in \mathbb{N}$ with $k \geq 2$ be a number. Then*

$$\int_{\partial R} \frac{1}{(z-a)^k} = 0\ .$$

Proof. Omitted for the lack of time but you can ponder it in Exercise 1 . $\square$

The Cauchy formulas are another important result in Complex Analysis. They express the value of a holomorphic function and its derivatives at a point by values in far away points, which demonstrates strange and fascinating non-locality of holomorphic functions. For simplicity we state and prove the formulas only for entire functions (and only the first derivative).

Theorem (two Cauchy formulae). *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and $a \in \mathbb{C}$. Then, with ρ $(= 2\pi i)$ being the above constant,*

$$f(a) = \frac{1}{\rho} \int \frac{f(z)}{z-a} \ \text{ and } \ f'(a) = \frac{1}{\rho} \int \frac{f(z)}{(z-a)^2}\ .$$

Proof. The existence of $f'(a)$ implies that $\frac{f(z)-f(a)}{z-a}$ is bounded on a deleted neighborhood of a. Thus by properties 1–3 of $\int$ we have

$$0 = \int \frac{f(z)-f(a)}{z-a} = \int \frac{f(z)}{z-a} - f(a) \int \frac{1}{z-a} = \int \frac{f(z)}{z-a} - f(a)\rho\ .$$

Since $\rho \neq 0$ by the proposition in the last lecture, the first Cauchy formula follows.

To prove the second one, for a given point $a \in \mathbb{C}$ we take a rectangle R with a inside.[1] For any point b inside R but $b \neq a$ we have by the first Cauchy formula and by the linearity of $\int_{\partial R}$ that

$$\begin{aligned} \frac{f(a)-f(b)}{a-b} &= \frac{1}{\rho}\int_{\partial R}\frac{f(z)}{(z-a)(z-b)} = \frac{1}{\rho}\int_{\partial R}\frac{f(z)}{(z-a)^2} + \\ &\quad + \frac{b-a}{\rho}\int_{\partial R}\frac{f(z)}{(z-a)^2(z-b)}\ . \end{aligned}$$

The ML bound shows that for every b close enough to a the last integral is in $|\cdot|$ bounded by a constant independent of b. Thus if b goes to a but is different from it, the left side goes to $f'(a)$ and the last term to 0, which gives the second Cauchy formula. $\square$

The proofs of Theorems 1–3. We prove Liouville's Theorem 2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire and bounded, $|f(z)| < c$ for every $z \in \mathbb{C}$ and a real constant $c > 0$. Let $a, b \in \mathbb{C}$ be two (distinct) points. By Exercise 2 it is easy to find for every $n \in \mathbb{N}$ a square S with side length $s \geq n$ and such that $a, b \in \mathrm{int}(S)$ and for every $z \in \partial S$ one even has that $|z-a|, |z-b| > \frac{s}{3} = \frac{\mathrm{per}(S)}{12}$. Then by the first Cauchy formula and linearity of $\int_{\partial R}$,

$$f(a)-f(b) = \frac{a-b}{\rho}\int_{\partial S}\frac{f(z)}{(z-a)(z-b)}\ .$$

The ML bound of this integral gives that in $|\cdot|$ it is at most

$$\frac{c}{\mathrm{per}(S)^2/144}\cdot \mathrm{per}(S) \leq \frac{36c}{n}$$

which for $n \to \infty$ goes to 0. Hence $f(a) = f(b)$ and f is a constant function.

To prove continuity of f' for any entire function $f\colon \mathbb{C} \to \mathbb{C}$ in Theorem 3 we use the second Cauchy formula: for a fixed $a \in \mathbb{C}$ and any $b \in \mathbb{C}$ the formula and the linearity of $\int_{\partial R}$ give that

$$\begin{aligned} f'(a) - f'(b) &= \frac{1}{\rho}\int_{\partial R}\frac{f(z)}{(z-a)^2} - \frac{1}{\rho}\int_{\partial R}\frac{f(z)}{(z-b)^2} \\ &= \frac{a-b}{\rho}\int_{\partial R}\frac{f(z)(2-a-b)}{(z-a)^2(z-b)^2} \end{aligned}$$

[1] I suppress the use of functional $\int$ from now. As I see it, its introduction was not really needed.

where R is any rectangle containing a and b in its interior. The ML bound shows that for every b close enough to a the last integral is in $|\cdot|$ bounded by a constant independent on b. Thus if b goes to a then $f'(b)$ goes to $f'(a)$ and f' is continuous at a.

Finally we prove Theorem 1 that every entire function $f\colon \mathbb{C} \to \mathbb{C}$ has a power series expansion (centered at 0). Let $a \in \mathbb{C}$ be arbitrary and R be a large enough rectangle such that $0, a \in \mathrm{int}(R)$ and for every $z \in \partial R$ one has $|a/z| = |a|/|z| < \frac{1}{2}$ and $|z-a| > 1$ (Exercise 3). Let $m \in \mathbb{N}$ be arbitrary. By means of the first Cauchy formula and the identity $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^m + \frac{x^{m+1}}{1-x}$ we get that

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\partial R}\frac{f(z)}{z-a} = \frac{1}{2\pi i}\int_{\partial R}\frac{f(z)}{z}\left(\sum_{n=0}^{m}(a/z)^n + \frac{(a/z)^{m+1}}{1-a/z}\right) \\ &= \sum_{n=0}^{m}\left(\frac{1}{2\pi i}\int_{\partial R}\frac{f(z)}{z^{n+1}}\right)a^n + \frac{1}{2\pi i}\int_{\partial R}\frac{f(z)(a/z)^{m+1}}{z-a} \\ &=: \sum_{n=0}^{m} c_n a^n + \frac{I_{m+1}}{2\pi i}\ . \end{aligned}$$

By the ML bound we are done: for $m \to \infty$,

$$|I_{m+1}| \le \max_{z\in\partial R}|f(z)| \cdot \frac{(1/2)^{m+1}}{1} \cdot \mathrm{per}(R) \to 0\ .$$

Thus for every $a \in \mathbb{C}$ we have

$$f(a) = \sum_{n=0}^{\infty} c_n a^n \ \text{ where } \ c_n = \frac{1}{2\pi i}\int_{\partial S}\frac{f(z)}{z^{n+1}}$$

with S being any rectangle with 0 inside.

Meromorphic functions, residues. We considerably generalize the proposition on ρ in the previous lecture. A set $A \subset \mathbb{C}$ is *discrete* if every ball $B(z,r) \subset \mathbb{C}$ contains only finitely many of its elements. A holomorphic function

$$f\colon U \setminus A \to \mathbb{C}\ ,$$

where the set $A \subset U$ is discrete, is a *meromorphic function* with the set of *poles* A if every point $a \in A$ has a neighborhood $U_a \subset U$ with $U_a \cap A = \{a\}$

such that for a holomorphic function $g_a\colon U_a \to \mathbb{C}$ and some numbers $k_a \in \mathbb{N}_0$ and $c_{j,a} \in \mathbb{C}$, $j = 1, 2, \ldots, k_a$, one has for every $z \in U_a \setminus \{a\}$ that

$$f(z) = g_a(z) + \sum_{j=1}^{k_a} \frac{c_{j,a}}{(z-a)^j}\ .$$

For $k_a = 0$ we define the sum as 0 ($f = g_a$ is then holomorphic on U_a). The coefficient $c_{1,a}$ is so called *residue of f at a*, denoted $\mathrm{res}(f, a) := c_{1,a}$. The first Cauchy formula implies that $\mathrm{res}(f, a)$ is uniquely determined by the function f (Exercise 4).

Theorem (on residues). *Let $f\colon U \setminus A \to \mathbb{C}$ be a meromorphic function with poles A and $R \subset U$ be a rectangle such that $\partial R \cap A = \emptyset$. Then the next sum is finite and*

$$\frac{1}{2\pi i}\int_{\partial R} f = \sum_{a \in A \cap \mathrm{int}(R)} \mathrm{res}(f,\, a) = \sum_{a \in A \cap R} \mathrm{res}(f,\, a)\ .$$

Thus the integral of f over ∂R, divided by $2\pi i$, equals to the sum of residues of f at the poles inside R.[2]

Proof. Infinitely many elements of A in $\mathrm{int}(R)$ would mean a limit point of A in R, contrary to the discreteness of A (Exercise 5). For every $a \in R \cap A$ we take a square $S_a \subset \mathrm{int}(R) \cap U_a$ centered at a and such that all these squares are disjoint. Then we split R in rectangles including all squares $\{S_a \mid a \in R \cap A\}$. We have

$$\begin{aligned}\int_{\partial R} f \;&=\; \sum_{a \in A \cap R} \int_{\partial S_a} f = \sum_{a \in A \cap R} \int_{\partial S_a} \left(g_a(z) + \sum_{j=1}^{k_a} \frac{c_{j,a}}{(z-a)^j} \right)\\ &=\; \sum_{a \in A \cap R} 2\pi i \cdot \mathrm{res}(f,\, a)\end{aligned}$$

and are done. The first equality follows by part 3 of the theorem on properties of $\int_u$ via an argument we applied already twice (Exercise 6). The second equality uses the definition of meromorphic functions. The third equality

[2] An advanced mathematical joke should be now understandable. Do you know that the contour integral of that function around the boundary of France is zero? ??? All Poles are in the eastern Europe!

follows from the linearity of integrals, the C.-G. theorem, the proposition on ρ, and the first proposition of this lecture. □

Generalization of the Basel problem. As an application of the theorem on residues and the complex analysis in general we determine the sum of the series

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}},\ \ k \in \mathbb{N}\ .$$

In lecture 10 we computed by a Fourier series that $\zeta(2) = \frac{\pi^2}{6}$.

Theorem ($\sum n^{-2k} =$?). *For every $k \in \mathbb{N}$ there is a positive fraction $\alpha_k \in \mathbb{Q}$ such that*

$$\zeta(2k) = \sum n^{-2k} = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \cdots = \alpha_k \pi^{2k}\ .$$

Proof. There exist fractions $B_0, B_1, \dots$, so called *Bernoulli numbers*, such that

$$\frac{x}{e^x - 1} = \sum_{r=0}^{\infty} \frac{B_r x^r}{r!}$$

(Exercise 7). We consider the meromorphic function $H\colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$,

$$H(z) = \frac{2\pi i}{e^{2\pi i z} - 1}\ ,$$

with the set of poles $\mathbb{Z}$ and the residue $\mathrm{res}(H, n) = 1$ for every $n \in \mathbb{Z}$ (Exercise 8). It is clear that if $f(z)$ is holomorphic on a neighborhood of $n \in \mathbb{Z}$ then $\mathrm{res}(fH, n) = f(n)$ (Exercise 12). We choose $f(z) = 1/z^{2k}$ and for $N \in \mathbb{N}$ denote by S_N the square with the vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. By the theorem on residues,

$$\frac{1}{2\pi i} \int_{\partial S_N} \frac{H(z)}{z^{2k}} = \sum_{n=-N}^{N} \mathrm{res}(H(z)z^{-2k},\, n) = \mathrm{res}(H(z)z^{-2k},\, 0) + 2\sum_{n=1}^{N} \frac{1}{n^{2k}}\ .$$

By Exercise 9 there is a constant $c > 0$ such that for every $N \in \mathbb{N}$ one has $z \in \partial S_N \Rightarrow |H(z)| < c$. By the ML bound the last integral is in $|\cdot|$ at most

$$\max_{z \in \partial S_N} \left| \frac{H(z)}{z^{2k}} \right| \cdot \mathrm{per}(S_N) < \frac{c}{N^{2k}} \cdot (8N + 4)\ ,$$

which for $N \to \infty$ goes to 0. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2} \operatorname{res}(H(z)z^{-2k}, 0) .$$

By the definition of the Bernoulli numbers we have

$$z^{-2k} H(z) = \frac{2\pi i \cdot z^{-2k}}{e^{2\pi i z} - 1} = \sum_{r=0}^{\infty} \frac{B_r (2\pi i)^r z^{r-1-2k}}{r!} .$$

Hence

$$\operatorname{res}(H(z)z^{-2k}, 0) = \frac{(-1)^k B_{2k} (2\pi)^{2k}}{(2k)!}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} B_{2k} \pi^{2k}$$

is a rational multiple of π^{2k}. $\square$

One can show that $B_{2k-1} = 0$ for $k \geq 2$ (Exercise 10). Further, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and so on (Exercise 11). The previous proof is taken from the book P. D. Lax and L. Zalcman, *Complex Proofs of Real Theorems*, AMS, Providence, RI, 2012.

Exercises

1. Prove that for every integer $k \geq 2$ and complex number a,

$$\int (z-a)^{-k} = 0 .$$

2. Construct for every $n \in \mathbb{N}$ and $a, b \in \mathbb{C}$ a square $S \subset \mathbb{C}$ with $s \geq n$, where s is the length of the side of S, and such that $a, b \in \operatorname{int}(S)$ and for every $z \in \partial S$ the distances $|z-a|, |z-b|$ are larger than $\frac{s}{3}$.

3. Show that for every $a \in \mathbb{C}$ there is a rectangle R such that $0, a \in \operatorname{int}(R)$ and for every $z \in \partial R$ one has $|a/z| < \frac{1}{2}$ and $|z-a| > 1$.

4. Why is the value of the residue of f at a uniquely determined by f?

5. Prove that every infinite subset of a rectangle R has in R a limit point.
6. Show how to split a rectangle R into sub-rectangles including the prescribed disjoint rectangles $R_1, R_2, \ldots, R_k \subset \mathrm{int}(R)$ so that the first equality in the proof of the residue theorem holds.
7. Prove that the Bernoulli numbers are fractions.
8. Prove that the function $\frac{2\pi i}{e^{2\pi i z}-1}$ has poles exactly in the set of integers and has all residues 1.
9. Show that this function is uniformly (in N) bounded on the boundaries of the squares S_N.
10. Prove that the Bernoulli numbers with odd indices > 1 are zero.
11. Is it true for the Bernoulli numbers that $\lim B_n = 0$?
12. Why for f holomorphic near $n \in \mathbb{Z}$ does one have $\mathrm{res}(fH, n) = f(n)$?