Lecture 12, December 19, 2019

The constant $\rho = 2\pi i$. The Cauchy–Goursat theorem. The functional $\int$

Several integrals. We are continuing in the proof of Theorems 1, 2, and 3 of the previous lecture. For $k \in \mathbb{N}$ and a segment u, the k-*equipartition* of u is the partition of u in k subsegments of the same length $\frac{1}{k}|u|$, which is the image of the partition $0 < \frac{1}{k} < \frac{2}{k} < \cdots < \frac{k-1}{k} < 1$ of the unit interval.

Proposition ($\int_{u,\,\partial R}$ of a lin. function). *Let $a, b, \alpha, \beta \in \mathbb{C}$ with $a \neq b$, $f(z) = \alpha z + \beta$, and R be a rectangle.*

1. *It is true that*
$$\int_{ab} f = \int_{ab} (\alpha z + \beta) = g(b) - g(a), \quad \text{where } \ g(z) = \alpha \frac{z^2}{2} + \beta z \ .$$

2. *Also,*
$$\int_{\partial R} f = \int_{\partial R} (\alpha z + \beta) = 0 \ .$$

Proof. 1. It is not hard to compute this as $\lim C(f, p_n)$ where the p_n are the n-equipartitions of ab. We leave it as Exercise 1.

2. Let the canonic vertices of R be a, b, c, d. By the definition of $\int_{\partial R}$ and part 1 we have
$$\int_{\partial R} f = g(b) - g(a) + g(c) - g(b) + g(d) - g(c) + g(a) - g(d) = 0 \ .$$
□

The proof of the next reduction of $\int_u f$ to the Riemann integral is left as Exercise 2.

Proposition ($\int_u$ and $(R)\int_a^b$). *Let $a, b \in \mathbb{C}$ with $a \neq b$, $f\colon ab \to \mathbb{C}$ be a continuous function, and $\varphi = t(b-a) + a\colon [0,1] \to \mathbb{C}$ be the parameterization defining the segment $u = ab$. Then*
$$\begin{aligned}\int_u f &= (R)\int_0^1 \mathrm{re}\left(f(\varphi(t)) \cdot \varphi'(t)\right)\,dt + i \cdot (R)\int_0^1 \mathrm{im}\left(f(\varphi(t)) \cdot \varphi'(t)\right)\,dt \\ &= (b-a)\left((R)\int_0^1 \mathrm{re}\left(f(\varphi(t))\right)\,dt + i \cdot (R)\int_0^1 \mathrm{im}\left(f(\varphi(t))\right)\,dt\right) .\end{aligned}$$

For completeness we mention the standard definition of the integral $\int_\varphi f$ of f along a curve φ which is a basic notion in complex analysis. If

$$f\colon U \to \mathbb{C}, \quad \text{and} \quad \varphi\colon [a,\, b] \to U$$

is a piecewise smooth and continuous function, then

$$\int_\varphi f := (R)\int_a^b \mathrm{re}\left(f(\varphi(t)) \cdot \varphi'(t)\right)\; dt + i \cdot (R)\int_a^b \mathrm{im}\left(f(\varphi(t)) \cdot \varphi'(t)\right)\; dt$$

if these Riemann integrals exist.

The next result is an under-appreciated pillar of complex analysis: if the constant ρ in it were 0, no Cauchy formulae, which we derive next time, would exist and the complex analysis would collapse.

Proposition (the constant $\rho = 2\pi i$). *Let S be the square with vertices $\pm 1 \pm i$. Then*

$$\rho := \int_{\partial S} \frac{1}{z} \neq 0, \quad \textit{in fact} \;\; \mathrm{im}(\rho) \geq 4\;.$$

Proof. The canonic vertices of S are $a = -1 - i$, $b = 1 - i$, $c = 1 + i$, and $d = -1 + i$. Let $p_n = (a_0, a_1, \ldots, a_n)$ be the n-equipartition of the segment ab. The multiplication by i geometricly means the rotation around 0 by the angle $\pi/2$ in the positive direction (counter-clockwisely). Thus $q_n := ip_n = (ia_0, ia_1, \ldots, ia_n)$ is the n-equipartition of bc. Similarly $r_n := iq_n = -p_n$ and $s_n := ir_n = -ip_n$ are the n-equipartitions of the segments cd and da, respectively. Surprisingly, for $f(z) = \frac{1}{z}$ we have

$$C(f,\, p_n) = C(f,\, q_n) = C(f,\, r_n) = C(f,\, s_n)\;.$$

Indeed, expanding the fraction by i we get

$$C(f,\, p_n) = \sum_{j=1}^{n} \frac{(b-a)/n}{a + j(b-a)/n} = \sum_{j=1}^{n} \frac{(ib-ia)/n}{ia + j(ib-ia)/n} = C(f,\, q_n)$$

because $ib = c$ and $ia = b$. The remaining two equalities follow in the same way. Since $b - a = 2$ and $a = -1 - i$, expanding the fraction by $\frac{2j}{n} - 1 + i$

we get

$$\begin{aligned}\operatorname{im}(C(f,\,p_n)) &= \operatorname{im}\left(\sum_{j=1}^{n}\frac{2/n}{-1-i+2j/n}\right) = \operatorname{im}\left(\frac{2}{n}\sum_{j=1}^{n}\frac{2j/n-1+i}{(2j/n-1)^2+1}\right)\\ &= \frac{2}{n}\sum_{j=1}^{n}\frac{1}{(2j/n-1)^2+1} \ge \frac{2}{n}\sum_{j=1}^{n}\frac{1}{2} = 1\ .\end{aligned}$$

Hence

$$\operatorname{im}(\rho) = \operatorname{im}\left(\int_{\partial S}\frac{1}{z}\right) = 4\cdot\operatorname{im}\left(\int_{ab}\frac{1}{z}\right) = 4\cdot\lim_{n\to\infty}\operatorname{im}(C(1/z,\,p_n)) \ge 4$$

(Exercise 3) and indeed $\rho \ne 0$. □

You can calculate in Exercise 5 that $\rho = 2\pi i$. This constant is ubiquitous in the complex analysis.

The Cauchy–Goursat theorem is in complex analysis result number 1: the integral $\int_\varphi f$ of a holomorphic function f over a simple closed curve φ (this means that φ is injective, except for $\varphi(a) = \varphi(b)$), which lies in the definition domain of f together with its interior, equals 0. But we only can integrate over boundaries of rectangles; complicated curves do not interest us.

For the proof of the theorem we need the notion of the *diameter* $\operatorname{diam}(X)$ of a set $X \subset \mathbb{C}$, and we remind an auxiliary result from the proof of the Baire theorem. We define

$$\operatorname{diam}(X) := \sup(\{|x-y| \mid x,\,y \in X\})\ .$$

The diameter may be $+\infty$. We leave the proof of the next proposition as Exercise 6, see also Exercises 7 and 8.

Proposition (nested closed sets). *If*

$$\mathbb{C} \supset A_1 \supset A_2 \supset \ldots$$

are nonempty and closed sets with $\lim \operatorname{diam}(A_n) = 0$*, then* $\bigcap_{n=1}^{\infty} A_n \ne \emptyset$*.*

We still need the construction of the *quarters* of a rectangle R with canonic vertices a, b, c, d. If $e = \frac{a+b}{2}$, $f = \frac{b+c}{2}$, $g = \frac{c+d}{2}$, and $h = \frac{d+a}{2}$ are the midpoints of the sides of R and $j = \frac{a+c}{2}$ is the midpoint of R, the four quarters of R are the rectangles A, B, C, and D with the canonic vertices

$$(a,\, e,\, j,\, h),\ (e,\, b,\, f,\, j),\ (j,\, f,\, c,\, g),\ \text{ and } \ (h,\, j,\, g,\, d)\ ,$$

respectively, to which is R split by cutting it along the segments eg a hf. For each of these quarters E, $\mathrm{per}(E) = \frac{1}{2}\mathrm{per}(R)$ and $\mathrm{diam}(E) = \frac{1}{2}\mathrm{diam}(R)$.

Theorem (Cauchy–Goursat). *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $R \subset U$ be a rectangle. Then*

$$\int_{\partial R} f = 0\ .$$

Proof. Let f, U, and R be as given. We construct such nested rectangles

$$R = R_0 \supset R_1 \supset R_2 \supset \dots$$

that for every $n \in \mathbb{N}_0$, R_{n+1} is a quarter of R_n and

$$\left| \int_{\partial R_{n+1}} f \right| \geq \frac{1}{4} \left| \int_{\partial R_n} f \right| .$$

Let $R_0, R_1, \dots, R_n$ have been already defined and A, B, C, and D be the quarters of the rectangle R_n. We claim that

$$\int_{\partial R_n} f = \int_{\partial A} f + \int_{\partial B} f + \int_{\partial C} f + \int_{\partial D} f\ .$$

This identity follows by applying part 3 of the theorem on properties of $\int$. Expanding each $\int_{\partial A} f, \dots, \int_{\partial D} f$ as a sum of four integrals over the sides yields 16 terms on the right side. Eight of them that correspond to the sides of the quarters lying inside R_n cancel out because they form four pairs of opposite orientations of four segments. The remaining eight terms that correspond to the sides of the quarters lying on ∂R_n sum up to the integral on the left side. The identity implies by the triangle inequality that there is a quarter $E \in \{A, B, C, D\}$ such that $|\int_{\partial E} f| \geq \frac{1}{4}|\int_{\partial R_n} f|$. We set $R_{n+1} = E$.

By the last proposition there exists a point z_0 such that

$$z_0 \in \bigcap_{n=0}^{\infty} R_n\ .$$

As $R_0 = R \subset U$, also $z_0 \in U$. Now we use the existence of the derivative $f'(z_0)$. For a given $\varepsilon > 0$ there is a $\delta > 0$ such that $B(z_0, \delta) \subset U$ and for some function $\Delta\colon B(z_0, \delta) \to \mathbb{C}$ we have for every $z \in B(z_0, \delta)$ that $|\Delta(z)| < \varepsilon$ (Exercise 9) and

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{g(z)} + \underbrace{\Delta(z)(z - z_0)}_{h(z)} \ .$$

We consider the marked functions $g(z)$ and $h(z)$. Clearly, $g(z)$ is linear and $h(z) = f(z) - g(z)$ is continuous. Suppose that $n \in \mathbb{N}_0$ is so big that $R_n \subset B(z_0, \delta)$. By part 2 of the first proposition (and by the linearity of the integral) we have

$$\int_{\partial R_n} f = \int_{\partial R_n} g + \int_{\partial R_n} h = \int_{\partial R_n} h \ .$$

By the ML bound (part 2 of the theorem on properties of $\int$),

$$\begin{aligned} \left| \int_{\partial R_n} h \right| &\le \max_{z \in \partial R_n} |\Delta(z)(z - z_0)| \cdot \mathrm{per}(R_n) \\ &< \varepsilon \cdot \mathrm{diam}(R_n) \cdot \mathrm{per}(R_n) = \varepsilon \cdot \frac{\mathrm{diam}(R)}{2^n} \cdot \frac{\mathrm{per}(R)}{2^n} \\ &< \varepsilon \cdot \frac{\mathrm{per}(R)^2}{4^n} \ . \end{aligned}$$

We used the above mentioned decrease for quarters of both the diameter and the perimeter to a half, and that the diameter of a rectangle is less than the perimeter. Thus

$$\frac{1}{4^n} \left| \int_{\partial R} f \right| \le \left| \int_{\partial R_n} f \right| = \left| \int_{\partial R_n} h \right| < \varepsilon \cdot \frac{\mathrm{per}(R)^2}{4^n}$$

and $| \int_{\partial R} f| < \varepsilon \cdot \mathrm{per}(R)^2$. This is true for every $\varepsilon > 0$ and $\int_{\partial R} f = 0$. □

It is a remarkable proof, isn't it? The author of the theorem is the French mathematician *Augustin-Louis Cauchy (1789–1857)* who during his political emigration lived in 1833 also in Prague. But Cauchy always assumed in his arguments that f' was continuous, and it was another French mathematician *Édouard Goursat (1858–1936)* who proved the theorem in 1900 only under the assumption of existence of f', in

E. Goursat, Sur la définition générale des fonctions analytiques, d'après Cauchy, *Trans. Amer. Math. Soc.* **1** (1900), 14–16.

The functional $\int$. We define for compact sets $A \subset \mathbb{C}$ — recall that these sets A are closed and bounded — the sets of holomorphic functions

$$H_A := \{f \colon \mathbb{C} \setminus A \to \mathbb{C} \mid f \text{ is holomorphic}\} \text{ and } H := \bigcup_{A \subset \mathbb{C} \text{ is compact}} H_A \ .$$

Thus H consists of the functions that are holomorphic on the complement of a compact set.

Definition (the functional $\int$). *We define $\int$, a function on the set H, by*

$$\int : H \to \mathbb{C}, \ \int f = \int_{\partial R} f \ ,$$

where $f \in H_A$ and R is an arbitrary rectangle such that $\mathrm{int}(R) \supset A$.

Before we state and in the next last lecture prove various properties of the functional $\int$, we show that its value is independent of the selection of the rectangle R and the definition is therefore correct.

For a function $f \in H_A$ and every two rectangles R and S with $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ we prove that

$$\int_{\partial R} f = \int_{\partial S} f \ .$$

Let first $S \subset \mathrm{int}(R)$. Extending the sides of S we split R in nine rectangles $R_1, \dots, R_8, S$. The same geometric argument as in the last proof gives the first of the next two equalities:

$$\int_{\partial R} f = \sum_{i=1}^{8} \int_{\partial R_i} f + \int_{\partial S} f = \int_{\partial S} f \ .$$

The second one, that always $\int_{\partial R_i} f = 0$, follows from the Cauchy–Goursat theorem because $R_i \subset \mathbb{C} \setminus A$. The general position of R and S reduces to this case. By Exercise 10 for every two rectangles R and S and every nonempty compact set A with $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ there exists a rectangle T such that

$$A \subset \mathrm{int}(T) \ \text{ and } \ T \subset \mathrm{int}(R) \cap \mathrm{int}(S) \ .$$

Here the property of rectangles that the intersection of every two of them, if their interiors intersect, is again a rectangle is useful. For discs or triangles this does not hold. Thus

$$\int_{\partial R} f = \int_{\partial T} f = \int_{\partial S} f \ .$$

Theorem (properties of the functional $\int$). *There are four important properties.*

1. *Linearity: for every $\alpha, \beta \in \mathbb{C}$ and $f, g \in H$,*

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g \ .$$

   But where is $\alpha f + \beta g$ defined? — Exercise 11.

2. *Extension of the C.–G. theorem: if $a \in \mathbb{C}$ and a function $f \in H_{\{a\}}$ is bounded on a deleted neighborhood of the point a, then*

$$\int f = 0 \ .$$

3. *Again ρ: for every $a \in \mathbb{C}$,*

$$\int \frac{1}{z-a} = \rho$$

   where ρ $(= 2\pi i)$ is the above constant.

4. *Exchange of a limit and $\int$: if $f, f_n \in H_A$, $n = 1, 2, \ldots,$ $A \subset \mathrm{int}(R)$ for a rectangle R, and $f_n \rightrightarrows f$ on ∂R, then*

$$\lim_{n\to\infty} \int f_n = \int f \ .$$

Exercises

1. Prove that $\int_{ab} (\alpha z + \beta) = \alpha(\frac{b^2}{2} - \frac{a^2}{2}) + \beta(b - a)$.

2. Prove the proposition reducing $\int_u$ to $(R) \int_a^b$.

3. Let (z_n) be a convergent sequence of complex numbers. Prove that $\mathrm{im}(\lim z_n) = \lim \mathrm{im}(z_n)$.

4. Prove that $\mathrm{re}(\rho) = 0$.

5. If $a = -1 - i$ and $b = 1 - i$, compute that

$$\int_{ab} \frac{1}{z} = \frac{\pi i}{2} \ . \ \text{ Hence } \ \rho = 4(\pi i/2) = 2\pi i \ .$$

6. Prove the proposition on nested closed sets in $\mathbb{C}$.

7. Prove that this proposition holds even if we only assume (instead of $\lim \mathrm{diam}(A_n) = 0$) that A_1 is bounded.

8. But show that this is not true in a general complete metric space.

9. What is the value at z_0 of the function $\Delta(z)$ in the proof of the C.–G. theorem?

10. Prove the lemma on the rectangle T in the proof of the correctness of the definition of $\int$.

11. If $f, g \in H$ and $\alpha, \beta \in \mathbb{C}$, where is $\alpha f + \beta g$ defined?

12. $\int 1/z^2 =?$