Lecture 10, December 5, 2019

Fourier series

Two more results on power series. Before F. series we mention without proofs two results on the limit behavior of functions given as sums of power series.

Proposition (nonnegative coefficients). *Let* $\sum_{n\ge 0} a_n x^n$ *be a power series with nonnegative real coefficients,* $c > 0$ *be a real number, and let us suppose that the given power series converges for every* $x \in [0, c)$*. Then the following limit and infinite sum always exist and are equal,*

$$\lim_{x\to c^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n c^n \ ,$$

no matter whether they are finite or $+\infty$*.*

Theorem (Abel's). *Let* $\sum_{n\ge 0} a_n x^n$ *be a power series with real coefficients,* $c > 0$ *be a real number, and let us suppose that the numeric series* $\sum_{n\ge 0} a_n c^n$ *converges. Then the given power series converges for every* $x \in [0, c)$*, the next limit exists, and one has the equality*

$$\lim_{x\to c^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n c^n \ .$$

The theorem bears the name of the Norwegian mathematician *Niels H. Abel (1802–1829).* Here is one application:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 \quad \text{(Exercise 1)} \ .$$

Fourier series. The rest of lecture 10 is a crash-course on Fourier series, no proofs. These series of functions bear the name of the French mathematician and physicist *Joseph Fourier (1768–1830).* A *trigonometric series* $F(x)$ is determined by real coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ and it is the series of functions

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \ .$$

The exceptional summand $\frac{a_0}{2}$ is explained in Exercise 5. In contrast with power series, trigonometric series may sum, as we will see, to discontinuous or non-smooth functions, which is their advantage. For two functions $f, g \in \mathcal{R}[-\pi, \pi]$ (i.e. f and g are Riemann-integrable on $[-\pi, \pi]$) we define the operation

$$\langle f,\, g\rangle := \int_{-\pi}^{\pi} fg\ .$$

It is almost a scalar product: it has the property of *symmetry*

$$\langle f,\, g\rangle = \langle g,\, f\rangle\ ,$$

the property of *bilinearity*

$$\langle \alpha f + \beta g,\, h\rangle = \alpha\langle f,\, g\rangle + \beta\langle g,\, h\rangle$$

($\alpha, \beta \in \mathbb{R}$ and $h \in \mathcal{R}[-\pi, \pi]$), and the property of *positive semidefinitness*

$$\langle f,\, f\rangle \ge 0$$

(Exercise 2). But in general the property $\langle f, f\rangle = 0 \Rightarrow f \equiv 0$ is missing (Exercise 3), so this operation is not a true scalar product.

Proposition (orthogonality of sines and cosines). *For every two integers $m, n \ge 0$,*

$$\langle \sin(mx),\, \cos(nx)\rangle = 0\ .$$

For every two integers $m, n \ge 0$, except $m = n = 0$,

$$\langle \sin(mx),\, \sin(nx)\rangle = \langle \cos(mx),\, \cos(nx)\rangle = \begin{cases} \pi & \dots \quad m = n \\ 0 & \dots \quad m \ne n\ . \end{cases}$$

Finally,

$$\langle \sin(0x),\, \sin(0x)\rangle = 0 \quad \textit{and} \quad \langle \cos(0x),\, \cos(0x)\rangle = 2\pi\ .$$

The Fourier series of a function. As we know (from *MA I*), one can associate with a function f, defined near 0 and with derivatives $f^{(n)}(0) \in \mathbb{R}$ of all orders $n = 0, 1, \dots,$ the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n\ ,$$

so called Taylor series of f. Often but not always it sums to $f(x)$. We similarly assign to a function $f \in \mathcal{R}[-\pi, \pi]$ a certain trigonometric series, so called Fourier series of f. In favorable circumstances it sums to $f(x)$.

For every function $f \in \mathcal{R}[-\pi, \pi]$ we first define its *cosine Fourier coefficients*

$$a_n = \frac{\langle f(x),\, \cos(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\; dx,\;\; n = 0,\, 1,\, \dots\;,$$

and its *sine Fourier coefficients*

$$b_n = \frac{\langle f(x),\, \sin(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\; dx,\;\; n = 1,\, 2,\, \dots$$

(Exercise 4). The trigonometric series

$$F_f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with these coefficients is the *Fourier series of the function f*. By means of $\langle \cdot, \cdot \rangle$, the previous proposition, and by exchanging summation and integration under uniform convergence one can prove the

Proposition (on Fourier coefficients). *Let $f \in \mathcal{R}[-\pi, \pi]$ and $a_0, a_1, \dots$ and $b_1, b_2, \dots$ be such real numbers that*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \rightrightarrows f(x) \quad (\text{on } [-\pi, \pi])\,.$$

Then a_n and b_n are the Fourier coefficients of the function f.

If a trigonometric series expresses a given (integrable) function f as its uniform sum, the coefficients in the series are inevitably the Fourier coefficients of f. Moreover, these coefficients satisfy the next inequality.

Theorem (the Bessel inequality). *If a_n and b_n are the Fourier coefficients of $f \in \mathcal{R}[-\pi, \pi]$ then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \le \frac{\langle f,\, f \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2\,.$$

The inequality is named after the German mathematician and astronomer *Friedrich W. Bessel (1784–1846).* It is not hard to deduce from it (in Exercise 6) the

Corollary (the Riemann–Lebesgue lemma). *For every function f in $\mathcal{R}[-\pi,\pi]$,*

$$\lim_{n\to\infty}\int_{-\pi}^{+\pi} f(x)\cos(nx)\;dx = \lim_{n\to\infty}\int_{-\pi}^{+\pi} f(x)\sin(nx)\;dx = 0\;.$$

Piecewise smooth functions. Roughly speaking, these are the functions which are sums of their F. series. A function $f\colon [a,b]\to\mathbb{R}$, where $a<b$ are real numbers, is *piecewise smooth* if there is a partition

$$a=a_0<a_1<a_2<\cdots<a_k=b,\;\; k\in\mathbb{N}\;,$$

of the interval $[a,b]$ such that on each interval (a_{i-1},a_i), $i=1,2,\dots,k$, the function f has continuous derivative f', for every $i=1,2,\dots,k$ there exist proper one-sided limits

$$f(a_i-0):=\lim_{x\to a_i^-}f(x)\;\;\text{and}\;\;f'(a_i-0):=\lim_{x\to a_i^-}f'(x)\;,$$

and for every $i=0,1,\dots,k-1$ there exist proper one-sided limits

$$f(a_i+0):=\lim_{x\to a_i^+}f(x)\;\;\text{and}\;\;f'(a_i+0):=\lim_{x\to a_i^+}f'(x)\;.$$

A piecewise smooth function may be discontinuous at a few points in the interval $[a,b]$, but in each point of discontinuity it has proper one-sided limits and it has in these points defined one-sided non-vertical tangents. There are many examples of piecewise smooth functions (Exercise 7).

We present (without proofs) two theorems on sums of Fourier series. From esthetic reasons we consider instead of functions $f\colon[-\pi,\pi)\to\mathbb{R}$ the 2π-*periodic functions* $f\colon\mathbb{R}\to\mathbb{R}$, which are the functions satisfying for every $x\in\mathbb{R}$ the equality $f(x+2\pi)=f(x)$. It is clear how the two kinds of functions mutually correspond (Exercise 8). A 2π-periodic function $f\colon\mathbb{R}\to\mathbb{R}$ is piecewise smooth if the restriction $f\,|\,[-\pi,\pi]$ is piecewise smooth.

Theorem (on $\rightrightarrows$ of a F. series). *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is 2π-periodic, continuous, and piecewise smooth. Then*

$$F_f(x) \rightrightarrows f(x) \ \ (\textit{on } \mathbb{R})\ .$$

Theorem (Dirichlet's on F. series). *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is 2π-periodic and piecewise smooth. Then*

$$F_f(x) \to \frac{f(x-0)+f(x+0)}{2} \ \ (\textit{on } \mathbb{R})\ .$$

If $x_0 \in \mathbb{R}$ is a point of discontinuity of the function f, the Fourier series of f has sum $F_f(x_0)$ equal to the arithmetic mean of the one-sided limits of f at x_0. If f is continuous at x_0 then $F_f(x_0) = f(x_0)$. *Peter G. L. Dirichlet (1805–1859)* was a German-French mathematician.

The Basel Problem. This was the problem to sum $\sum 1/n^2$. It was resolved by *Leonhard Euler (1707–1783)* who in fact was born in Basel. We obtain this sum by expanding the function

$$f\colon [-\pi, \pi] \to \mathbb{R},\ f(x) = x^2\ ,$$

in Fourier series. Since $f(-\pi) = f(\pi)$, we can regard f as a 2π-periodic and everywhere defined function $f\colon \mathbb{R} \to \mathbb{R}$. It is an even function; in general a function $g\colon \mathbb{R} \to \mathbb{R}$ is *even* if for every $x \in \mathbb{R}$, $g(-x) = g(x)$. A function $g\colon \mathbb{R} \to \mathbb{R}$ is *odd* if for every $x \in \mathbb{R}$, $g(-x) = -g(x)$. Let $a > 0$ be a real number and $g \in \mathcal{R}[-a, a]$. Then we have the identities

$$g \text{ is even} \Rightarrow \int_{-a}^{a} g = 2\int_0^a g \ \text{ and } g \text{ is odd} \Rightarrow \int_{-a}^{a} g = 0 \ \text{ (Exercise 9)}\ .$$

Back to the Fourier expansion of our 2π-periodic parabola $f(x) = x^2$. By the last identity and Exercise 10, all sine Fourier coefficients b_n of f are zero. It suffices to calculate its cosine Fourier coefficients a_n. By their definition, the last but one identity, and Exercise 10,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2\pi^2}{3}\ ,$$

and for $n \in \mathbb{N}$ by two integrations by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x^2 \overbrace{\cos(nx)}^{(\sin(nx)/n)'}\,dx = \frac{2}{\pi n}\underbrace{\left[x^2\sin(nx)\right]_0^\pi}_{0-0=0} - \frac{4}{\pi n}\int_0^\pi x\ \overbrace{\sin(nx)}^{(-\cos(nx)/n)'}\ dx \\ &= \frac{4}{\pi n^2}\underbrace{\left[x\cos(nx)\right]_0^\pi}_{\pi(-1)^n} - \frac{4}{\pi n^2}\underbrace{\int_0^\pi \cos(nx)\,dx}_{0-0=0} = (-1)^n\frac{4}{n^2}\ . \end{aligned}$$

Since $f(x)$ is continuous and piecewise smooth, by the penultimate theorem we have that

$$F_f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2} \rightrightarrows f(x)\ \ (\text{on } \mathbb{R})\ .$$

For $x = \pi$ we get

$$\pi^2 = f(\pi) = F_f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n^2},\quad \text{hence}\quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\ .$$

Exercises

1. Prove with the help of Abel's theorem that $\sum_{n\ge1}(-1)^{n+1}/n = \log 2$.
2. Prove the symmetry, bilinearity, and positive semidefinitness of the operation $\langle\cdot,\cdot\rangle$.
3. Give an example of a non-zero function $f \in \mathcal{R}[-\pi,\pi]$ with $\langle f, f\rangle = 0$.
4. Why are the cosine and sine Fourier coefficients of a function $f \in \mathcal{R}[-\pi,\pi]$ correctly defined?
5. Why is the constant coefficient in trigonometric series $\frac{a_0}{2}$ and not a_0?
6. Deduce the Riemann–Lebesgue lemma from the Bessel inequality.
7. Is a broken line a piecewise smooth function?

8. Let

$$M_1 = \{f \mid f\colon [-\pi, \pi) \to \mathbb{R}\} \ \text{ and } \ M_2 = \{f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ is } 2\pi\text{-per.}\}\ .$$

Describe a natural bijection between the sets M_1 a M_2.

9. Prove the stated identities for the integral of an even, resp. odd, function over a symmetric interval.

10. Prove that the product of two even and two odd functions is an even function, and that the product of an even and an odd function is an odd function.

11. Expand in Fourier series the function $f(x) = \pi - x\colon [-\pi, \pi) \to \mathbb{R}$ and by an appropriate specialization deduce the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.