

Speeds of permutation classes

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(0000000000000000)

A) Growth functions in general

B) Top permutation growths

C) Bottom — — —

D) Forbidden monotone subsequences

curvilinear

A) Growth functions in general

(P, \leq) - poset of combinatorial structures

$$FCP, \overrightarrow{Av(F)} = \{x \in P \mid x \not\geq y \forall y \in F\}.$$

Structures avoiding every $y \in F$.

$$I \subset P \text{ is an ideal} : \underline{x \leq y \in I} : \Rightarrow x \in I.$$

lower ideal, downward closed set.

Every $Av(F)$ is an ideal.

• Every ideal I has form $I = Av(F)$ for some $F \subset P$.

In fact, F may be taken to be an anti-chain.

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$| \cdot | : P \rightarrow N = \{0, 1, 2, \dots\}$ - size function

$$S \subset P, S_n = \overline{\{x \in S \mid |x| = n\}}$$

Assume every P_n is finite.

$n \mapsto |S_n|$ - growth (function)
of the set S .

$S = I$ -ideal. What can be said,
integers of exact formulas
of asymptotic \sim) about

the growth functions $n \mapsto |T_n|$?

Some examples:

Heredity graph properties

$$P = \{G = ([n], E) \mid n \in \mathbb{N}\}, [n] = \{1, 2, \dots, n\}.$$

\leq ... induced subgraph relation



$|G| = n = \#$ of vertices.

Thm:

(Balogh, Bollobás & Weinreich,
hereditary gr. property 2000)

$T \subset P$ an ideal s.t. $|T_n| \subset C^n$ $\forall n$.

then $|T_n| = \sum_{i=1}^2 p_i(n)$ (for $n > n_0$)

$p_i(x)$ - polynomials. In other words,

$$\sum_{n \geq 0} |T_n| x^n = \frac{\text{poly}(x)}{(1-x)(1-2x)\dots(1-p_X)}.$$

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Ideals of permutations

$$\mathcal{P} = \{a_1 a_2 \dots a_n \mid \{a_1, \dots, a_n\} = \{1, \dots, n\}, \\ n \in \mathbb{N}\}.$$

\subset - Containment of permutations:

$$\pi = a_1 a_2 \dots a_m \subset g = b_1 b_2 \dots b_n \text{ if}$$

\exists subseq: $b_{i_1} b_{i_2} \dots b_{i_m}$ s.t. $a_{i_r} < a_s \Leftrightarrow b_{i_r} < b_{i_s}$.

$$\text{E.g.: } 132 \subset 4672351.$$

$$|a_1 a_2 \dots a_m| = m = \text{length of } a_1 \dots a_m$$

\mathcal{I} ... an ideal of permutations.

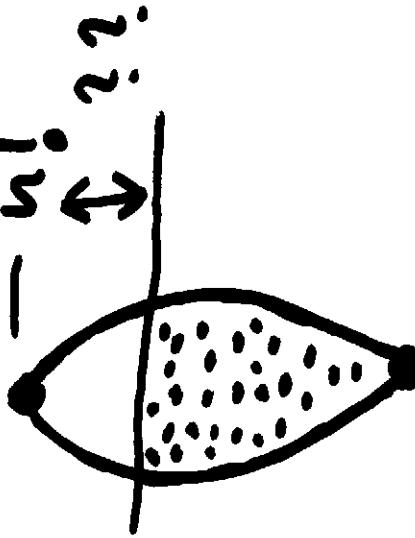
$$h \mapsto |I_h| \geq 2.$$

B) Top permutation growths

P = all permutations.

$T = P$, then $|T_n| = n!$.

But what if $T \neq P$, that is,
 $T \subset \text{Av}(\pi)$ for some $\pi \in P$?



How much must
the growth drop from
 $n!$?

$\equiv 0$ for $n > n_0$.

Stanley-Wilf conjecture, 1997 for every π .

Permutation growths should drop
from $n!$ to exponential growth.

Theorem (Marcus & Tardos, 2004)

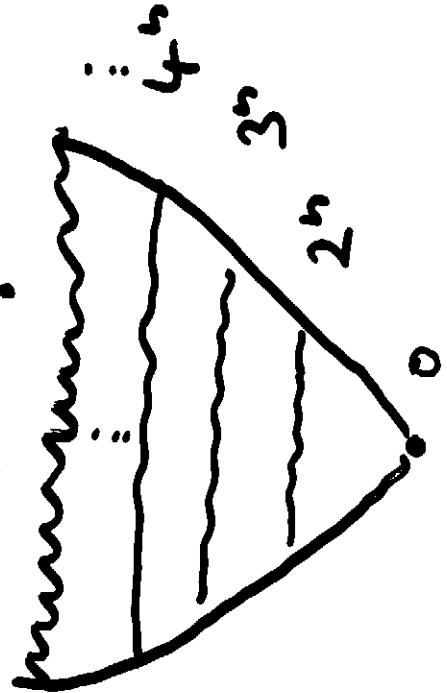
Indeed $|Av_n(\pi)| < c^n H\pi$.

Proof (sketch)

1. # 1's in an $n \times n$ 0-1 matrix H avoiding a fixed perm. sub-matrix $H\pi$ is $\mathcal{O}(n)$.
2. An inductive argument using bound 1 gives that $\# \text{ } n \times n$ 0-1 matrices H avoiding $H\pi$ is $< c^n$. \blacksquare

So we have this picture

$$\cdot n!$$



Theorem (Arzela 199, R. T. 104)

$\lim_{n \rightarrow \infty} |\text{Av}_n(\pi)|^{1/n}$ exists and is finite for every π .

Proof. Can assume: $\pi \neq \square$.

Thus

$$\pi \neq \square, \quad \Rightarrow \pi \notin \bigcup_{m=n}^{\infty} \square,$$

$$\text{So } |\text{Av}_n(\pi)| \cdot |\text{Av}_n(\pi)|^{1/n} \leq |\text{Av}_{m+n}(\pi)|$$

and $\lim_{n \rightarrow \infty} |\text{Av}_n(\pi)|^{1/n}$ exists (Fekete's lemma).

Cannot be $+\infty$ by Th. 9.T.



Problem Does $\lim_{n \rightarrow \infty} |\text{Av}_n(F)|^{1/n}$ exist for every set F ?

?

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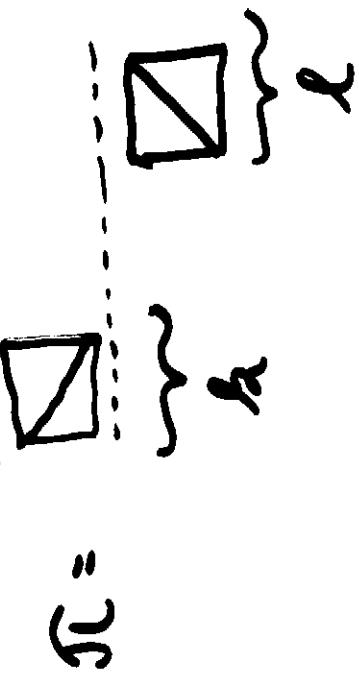
Examples

- $\pi = 123 \dots k = i_k \cdot \lim_{n \rightarrow \infty} |Av_n(i_k)|^{1/n} = (2-1)^2.$

- $|Av_n(132)| = \frac{1}{n+1} \binom{2^n}{n} \dots \text{Catalan numbers } \approx 4^n.$

Proof. $\pi \in Av_n(132)$ have form

$n = \dots$ where $\alpha + \ell = n - 1$



and $\pi \in Av_\alpha(132), \boxed{\pi \in Av_\ell(132)}$ are arbitrary. So for $c_n = |Av_n(132)|$

we have $c_n = \sum_{\alpha=0}^{n-1} c_\alpha \cdot c_{n-1-\alpha}$ among Catalan numbers. $(c_0 = 1)$ \(\blacksquare\)

- Schmidt & Simion, 185, determined every $\pi \mapsto |\text{Av}_n(\pi)|$ for $F \subset P_3$ ($2^6 = 64$ problems). In particular,
 $|\text{Av}_n(\pi)| = \frac{1}{n+1} \binom{2n}{n}$ if $\pi \in P_3$.
- What about $|\text{Av}_n(\pi)|$ for $\pi \in P_4$?
Stanová, West, 194: 24 problems split in 3 families:
 $|\text{Av}_n(1342)| = 1, 2, 6, 23, 103, 512, 2740, \dots$
 $|\text{Av}_n(1234)| = \dots$
 $|\text{Av}_n(1324)| = \dots$
- Theorem (Bóna, 197) Let $a_n = |\text{Av}_n(1342)|$.
Then $\sum_{n \geq 0} a_n x^n = \frac{(1-8x)^{3/2} - 8x^2 + 20x + 1}{2(x+1)^3}$.
- In particular, $|\text{Av}_n(1342)| \xrightarrow{n \rightarrow \infty} 8$.

- We know that $|\text{Av}_n(1234)| \xrightarrow{n \rightarrow \infty} 9$.

Theorem (Bessel) 1901, 197.

$$\text{Av}_n(1234) = 2 \sum_{q=0}^n \binom{n}{2q} \binom{q}{2} \frac{3q^2 + 2q + 1 - n - 2q^2}{(2+1)^2(2+2)(n-q+1)}.$$

$$= \frac{1}{(n+1)^2(n+2)} \sum_{q=0}^n \binom{n}{q} \binom{n+1}{q} \binom{n+2}{q+1}.$$

(we return to this in part D)

Problem Determine $|\text{Av}_n(1324)|$.

Not even the growth constant is known.
But it is known that this growth is the fastest of the three.

Theorem (Albert, Elder, Rechnitzer,
Westcott & Zabrocki) 2006

$$\lim_{n \rightarrow \infty} |\text{Av}_n(1324)| \xrightarrow{n \rightarrow \infty} 9.47.$$

$11 < \cdot < 12 \ ?$

• Bóna (105) found examples of non-integral growth constants:

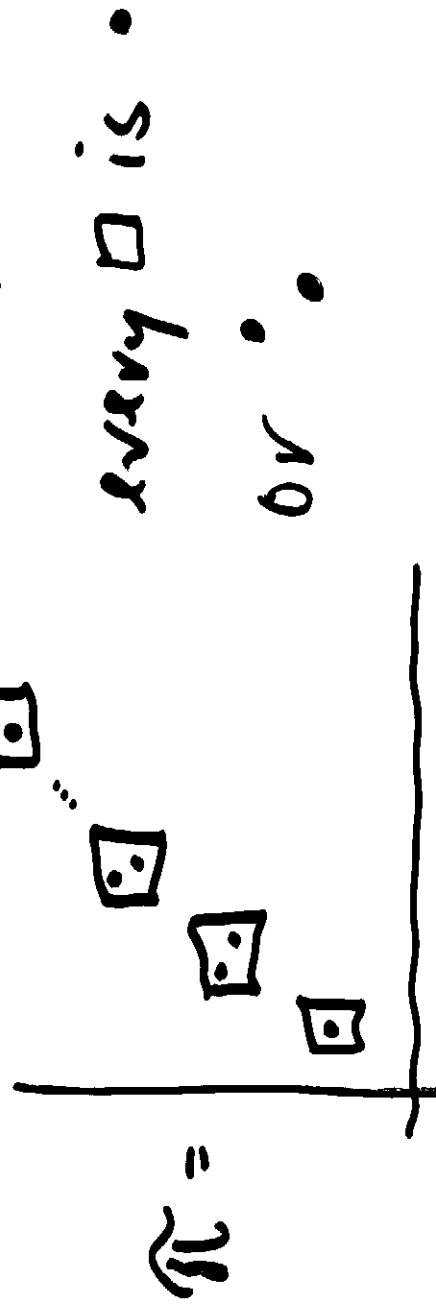
$$|Av_n(12453)|^{1/n} \rightarrow q + 4\sqrt{2}.$$

mmmmmmmmmmmmmm

c) Bottom permutation grows

Example

\mathcal{T} = all permutations of the form



then $|\mathcal{T}_n| = \#(b_1, b_2, \dots, b_n)$,

$$b_i \in \{1, 2\}, \quad \sum b_i = n.$$

$$\text{So } |\mathcal{T}_n| = F_n, \quad F_n = (1, 2, 3, 5, 8, 13, \dots)$$

Fibonacci numbers

Kaisera K1.) 'D3 : $(F_n)_{n \geq 1}$ is the slowest exp. growth. More precisely,

$I \subset P$ ideal $\Rightarrow |I_n| \geq F_n$ for $n \in \mathbb{N}$ or

$|I_n| = \text{poly}(n)$ for $n > n_0$.

We want about π of the form

$$\pi = \frac{\dots}{\boxed{\dots} \quad \boxed{\dots} \quad \boxed{\dots}}$$

$F_{n,2} := \#(b_1, b_2, \dots, b_r), b_i \in \{1, 2, \dots, 2^r\},$
 $\sum b_i = n$... generalized Fib. numbers
 $(\sum_{k \geq 0} F_{n,2} x^n = 1 / ((1-x-x^2-\dots-x^{2^r})).$

$F_{n,1} = 1, F_{n,2} = F_n, F_{n,2} \sim 1.61^n$,

$F_{n,2} \sim c_2^n, c_2 \uparrow 2$ for $n \rightarrow \infty$.

Theorem (Kaiser & K., '03)

Let ΓCP be an ideal of $Perv_{\mathcal{M}}$.
 Then exactly one of (i)-(iii) holds.

- (i) $|T_n| = \text{poly}(n)$ for $n > n_0$.
(ii) $\exists c_1, \varrho \in \mathbb{N}, \varrho \geq 2$ such that

$$F_{n,2} \leq |T_n| \leq h^c \cdot F_{n,2}$$

(iii) $|T_n| \geq 2^{n-1} \neq n$.

(Generalized by Balogh, Bollobás & Morris,
2006, to ordered graphs)

A diagram illustrating a sequence or pattern. On the left, there is a vertical column of 11 horizontal tick marks. To the right of this column is a vertical line. Above the vertical line, the expression 2^{n-1} is written. To the right of the vertical line, there are three dots (...), indicating that the sequence continues.

Fibonacci

evalu^hchy

F₅⁴ F₅³

poly(

What is going on above 2^{n-1} ?
Until recently not known but

Vetter, 107 (manuscript) has succeeded in describing gourds up to (2.20557...). In particular,

The first exp. growth a boric acid is
root of $1+2x^2-x^3$

(2. 06599...) $\sqrt{}$ root of $1 + x - x^2 - 2x^4 + x^5$.

A vertical black spiral binding is visible along the left edge of a page. The spiral is made of a thick, textured wire that forms a continuous loop from top to bottom.

Hairin problem for $F_C P$, greater

function $n \mapsto |Av_n(F)|$, find

- effective algorithms
 - asymptotic complexities

Some limitations:

(, There are uncountably many growths)
 $n \mapsto |Av_n(F)|.$

Proof. Let $A = \{\pi_1, \pi_2, \dots\}$ CP be
an infinite antichain s.t. $|\pi_1| < |\pi_2| < \dots$
... Then all growths $n \mapsto |Av_n(F)|$,
 $F \subset A$, are distinct. ⊗

thus almost all growths are not even computable. But what if F is finite?

Recall that $(a_n) \subset \mathbb{C}$ is holonomic
(D-finite, P-recursive) if

$$\sum_{n \geq 0} p_n(n) a_n + \sum_{n \geq 1} p_{n-1}(n) a_{n-1} = 0$$

where $p_i(x) \in \mathbb{C}[x]$
are polynomials
Hence.

All explicitly known $n \mapsto |Av_n(F)|$ are holonomic but, by , almost all growths are not.

Conjecture $n \mapsto |Av_n(1324)|$ is not holonomic.

(Or give another explicit example.)
~~consequently nonholonomic~~

d) $n \mapsto |Av_n(123\dots n)|$ - Forbidden

monotone subsequences

We have seen that $|Av_n(123)| = \frac{1}{n+1} \binom{2n}{n}$ ($|Av_n(12)| = 1$, $|Av_n(11)| = 0$) and the two formulas for $|Av_n(1234)|$.

In general,

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Bessel, 1990: When the growth
 $n \mapsto |Av_n(123\dots n)|$ is holonomic.

Let, for $i \in \mathbb{Z}_+$,

$$J_i(2x) = \sum_{n \geq 0} \frac{x^{2n+i}}{n!(n+i)!} \dots \text{ hyperbolic Bessel function}$$

of the 1st kind of order i .

Theorem (G.) 1990

$$\sum_{n \geq 0} |Av_n(123\dots (2+i))| x^n f(n!)^2 =$$

$$= \det(J_{i-j}(2x))_{i,j=1}^n$$

Example • $\sum_{n \geq 0} |Av_n(123)| x^n = |J_0(2x)|^2 =$

$$= \sum_{n \geq 0} \frac{\Delta_{n+1}(v)}{(n!)^2} x^n / (n!)^2.$$

• $a_n := |Av_n(12345)|$. Then $(n+4)(n+3) a_n =$

$$= (20n^3 + 62n^2 + 22n + 24)a_{n-1} - 64n(n-1)^2 a_{n-2}.$$

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For much more info on this topic:

Stanley, inv. lecture (plenary) on Tech'06.

Problem Holonomicity of $\prod_{k=1}^n |Av_n(c_k)|$
via D6F $\sum |Av_n(c_k)| x^n$ with $c_k = \frac{1}{k!} \prod_{i=1}^{k-1} i^{c_i}$

new variables?

$z_1 = 1, \dots, z_n$... easy

$z_{n+1} = 4, \dots, Bouquet - Nélus, '03.$

z_1, z_2, \dots, z_n ? ?

Finally, the asymptotics is

Theorem (Regev, 1981)

$$|Av_n(1, 2, \dots, (n+1))| \sim c_n \cdot \frac{\pi^{2n}}{n^{(n^2-n)/2}}$$

where

$$c_n = \left(\frac{1}{\sqrt{2\pi}} \right)^n \cdot \left(\frac{n}{2} \right)^{\frac{n(n-1)}{2}} \cdot \prod_{i=1}^n i!$$