Counting Even and Odd Partitions

Martin Klazar

1. INTRODUCTION. It is a lovely fact that $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$, has as many subsets X of even cardinality $|X|$ as of odd cardinality, namely, 2^{n-1} of both. To prove it, pair every subset X with $X \pm 1$, where $X \pm 1$ is $X \backslash \{1\}$ if $1 \in X$ and $X \cup \{1\}$ if $1 \notin X$. Then $X \mapsto X \pm 1$ is an involution that changes the parity of $|X|$ and the result follows.

More generally, in enumerative combinatorics one often has a family $\mathcal{S}_n$ of objects on $[n]$ such that every object X has a natural *size* $s(X)$ in $\mathbf{N}_0$. Then in addition to the total number of objects $S_n = |\mathcal{S}_n|$, one can consider

$$S_n^{\pm} = \sum_{X \in \mathcal{S}_n} (-1)^{s(X)},$$

which records the surplus of the objects with an even size over those with an odd size. For subsets X of $[n]$ and $s(X) = |X|$, it is the case that $S_n^{\pm} = 0$ for every $n \geq 1$ (but $S_0^{\pm} = 1$). In this note we present to the reader four examples of the scenario under discussion. We investigate the corresponding numbers $S_n^{\pm}$ by means of generating functions, an analytic continuation argument, and, again, the involution trick. Our first example is a classic, but the other three are not as well known.

2. INTEGER PARTITIONS. Here $\mathcal{S}_n$ consists of the partitions X of n into distinct parts — $n = a_1 + a_2 + \cdots + a_k$, where $a_1 > a_2 > \ldots > a_k \geq 1$ are integers — and $s(X) = k$ is just the number of parts.

Theorem 1 (L. Euler, 1748). *For integer partitions with distinct parts, $S_n^{\pm} = (-1)^m$ if $n = m(3m \pm 1)/2$ and $S_n^{\pm} = 0$ otherwise.*

This is Euler's celebrated pentagonal identity, which can be written equivalently as

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(3m+1)/2}.$$

Franklin's famous 1881 proof using the involution trick is reproduced in the books of Andrews [1] and Hardy and Wright [5].

3. NONCROSSING SET PARTITIONS. A (set) partition of $[n]$ is a collection $X = \{B_1, B_2, \ldots, B_k\}$ of nonempty disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. It is *crossing* if there are four numbers $1 \le a < b < c < d \le n$ and two distinct blocks A and B in X such that a and c belong to A, while b and d belong to B. If X is not crossing, then it is *noncrossing*. In this example, $\mathcal{S}_n$ consists of the noncrossing partitions of $[n]$ and $s(X) = k$ is the number of blocks. Kreweras [6] proved that $S_n = |\mathcal{S}_n| = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. The survey [11] of Simion contains much information on the combinatorics of noncrossing partitions.

Theorem 2. *For noncrossing set partitions, $S_n^{\pm} = (-1)^{m+1}\frac{1}{m+1}\binom{2m}{m}$ if $n = 2m+1$ and $S_n^{\pm} = 0$ if $n = 2m$.*

Proof. Let

$$F = F(x, y) = \sum_{n \ge 0} \sum_{X \in \mathcal{S}_n} x^n y^{s(X)} = 1 + xy + x^2(y + y^2) + \cdots .$$

We are interested in

$$G = G(x) = \sum_{n \ge 0} S_n^{\pm} x^n.$$

Clearly, $G(x) = F(x, -1)$. We show that

$$F = 1 + xyF + xF(F - 1). \tag{1}$$

The empty partition X of $[0] = \emptyset$ is represented by the term $x^0 y^0 = 1$. Now let X be a noncrossing partition of $[n]$, where $n \ge 1$, and let A be the block of X containing 1. Either $|A| = 1$ or $|A| > 1$. In the former case, $A = \{1\}$, and the removal of A (the remaining vertices are relabelled as $1, 2, \ldots, n-1$) constitutes a bijection between the noncrossing partitions X of $[n]$ with $|A| = 1$ and $s(X) = k$ and all noncrossing partitions Y of $[n-1]$ with $s(Y) = k-1$. Thus the case $|A| = 1$ is accounted for by the middle term xyF. In the case $|A| > 1$, we let a denote the second element of A and decompose X into two partitions X_1 and X_2, where X_1 is induced by X on the interval $[2, a-1]$ and X_2 is induced on $[a, n]$. Both X_i are noncrossing. The collection X_1 may be empty, but X_2 is nonempty. Since no block intersects both intervals (X is noncrossing), $s(X_1) + s(X_2) = s(X)$. The mapping $X \mapsto (X_1, X_2)$ (the vertices in X_1 and X_2 are relabelled appropriately) constitutes a bijection between the noncrossing partitions X of $[n]$ with $|A| > 1$ and $s(X) = k$ and the pairs (X_1, X_2) such that X_i is a noncrossing partition of $[n_i]$, $n_1 \geq 0$, $n_2 \geq 1$, $n_1 + n_2 = n - 1$, and $s(X_1) + s(X_2) = k$. Thus the case $|A| > 1$ is captured by the last term $xF(F-1)$.

Setting $y = -1$ in (1) and rearranging, we get the equation

$$xG^2 - (1+2x)G + 1 = 0.$$

Because $G(0) = 1$, we solve to obtain

$$G(x) = 1 + \frac{1}{2x}\Big(1 - \sqrt{1+4x^2}\Big).$$

We think of $G(x)$ as a formal power series and therefore $x = 0$ causes no problem. Binomial expansion yields the stated formula for $S_n^{\pm}$. Note that, by setting $y = 1$ in (1), we recover the result of Kreweras. □

One may ask about a proof using involutions. Such a proof, based on the representation of noncrossing partitions by parallelogram polyominoes, was provided by the referee. See Deutsch [3, pp. 198–199] for a bijection between noncrossing partitions and parallelogram polyominoes.

4. ALL SET PARTITIONS. Now $\mathcal{S}_n$ consists of all partitions of $[n]$ and $s(X) = k$ is again the number of blocks. The total numbers S_n are the *Bell numbers*

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, \ldots$$

that constitute sequence A000110 of [12]. They grow superexponentially:

$$\log S_n = n(\log n - \log\log n + O(1)).$$

See de Bruijn [2, p. 108] or Lovász [7, Problem 1.9b] for more precise asymptotics. We show that $S_n^{\pm}$ remains superexponential.

Theorem 3. *For all set partitions, if $c > 0$ is any constant, then $|S_n^{\pm}| > c^n$ for some (in fact, infinitely many) n in $\mathbf{N}$.*

Proof. We begin with the classical expansion (see, for example, Stanley [13, p. 34])

$$G_k(x) = \sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)},$$

where $S(n,k)$, the Stirling number of the second kind, is in our language simply the number of X in $\mathcal{S}_n$ with $s(X) = k$ blocks. Thus

$$F(x) = \sum_{n\geq 0} S_n^{\pm} x^n = \sum_{k\geq 0} (-1)^k G_k(x) = \sum_{k\geq 0} \frac{(-x)^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Considering the action of the substitution $x \mapsto x/(1-x)$ on this expansion, we obtain the equation

$$F(x) = 1 - \frac{x}{1-x}\, F\left(\frac{x}{1-x}\right). \tag{2}$$

Substituting $x/(1+x)$ for x and solving the resulting equation for $F(x)$, we arrive at a second expression for $F(x)$:

$$F(x) = \frac{1}{x}\left(1 - F\left(\frac{x}{1+x}\right)\right). \tag{3}$$

If $|S_n^{\pm}| \leq c^n$ for all n in $\mathbf{N}$ and some constant $c > 0$, then the power series representing $F(x)$ has radius of convergence $r \geq 1/c > 0$ and therefore defines in the disc $|z| < r$ an analytic function $F(z)$. However, we show that $r > 0$ is contradicted by the equations (2) and (3). Thus for no $c > 0$ is it true that $|S_n^{\pm}| \leq c^n$ for all n, and Theorem 3 follows.

Suppose, to the contrary, that $r > 0$. We can assume that $r \leq 1$ (formulas (2) or (3) show that $F(x)$ is not a polynomial, so $|S_n^{\pm}| \geq 1$ infinitely often). Let α be a singularity of $F(z)$ on the circle of convergence $|z| = r$. If $|\alpha/(1-\alpha)| < r$, we can use (2) to continue $F(z)$ analytically to a neighborhood of α, which contradicts the definition of α. Clearly, $|\alpha/(1-\alpha)| < r$ is equivalent to $\mathrm{Re}(\alpha) < r^2/2$, and therefore when $\mathrm{Re}(\alpha) < r^2/2$ we have derived a contradiction. Similarly, if $|\alpha/(1+\alpha)| < r$, which is equivalent to $\mathrm{Re}(\alpha) > -r^2/2$, we use (3) to obtain the same contradiction. (Since $\alpha \neq 1$ in the former case, $\alpha \neq -1$ in the latter case, and $\alpha \neq 0$ in every case, the "bad" arguments $z = -1, 0$, and 1 cause no problem.) For every location of α, either (2) or (3) leads to a contradiction. Hence $r = 0$. $\square$

The numbers $S_n^{\pm}$ that arise in this example,

$$-1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, -50533, 110176, \ldots,$$

comprise sequence A000587 of [12]. Recently the asymptotics of this sequence were investigated by Subbarao and Verma [15] and Yang [17] (see [12] for more references to these numbers). Subbarao and Verma proved, using the exponential generating function of $S_n^{\pm}$, that in fact

$$\limsup_{n\to\infty} \frac{\log |S_n^{\pm}|}{n \log n} = 1.$$

Is $S_n^{\pm}$ zero infinitely often? In [17], this question is atributed to H. S. Wilf. Is $S_n^{\pm}$ ever zero except when $n = 2$?

5. MATCHINGS AND CROSSINGS. Perhaps the lack of cancellation in the previous example was caused by the rapid growth of S_n? Our last example shows that $S_n^{\pm}$ can be small even if the S_n are superexponential. For it we take $\mathcal{S}_n$ to be all partitions X of $[2n]$ into n two-element blocks. We call such X *matchings* and their blocks *edges*. The size $s(X)$ is the number of unordered crossing pairs $\{A, B\}$ among the edges of X (we have defined crossing in the second example). It is easy to see that $S_n = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$. Indeed, $S_n = (2n-1)S_{n-1}$ because there are $2n-1$ ways to place the end of the new first edge in the spaces of an X from $\mathcal{S}_{n-1}$. So $\log S_n = n(\log n + O(1))$. But the $S_n^{\pm}$ are very small.

Theorem 4. *For matchings whose size is measured by the number of crossings, $S_n^{\pm} = 1$ for every n in $\mathbf{N}$.*

Proof. For a matching X in $\mathcal{S}_n$ the *crucial pair* is the pair of edges A and B in X such that $\min A + 1 = \min B$ and $\min A$ is as small as possible. Notice that the crucial pair is unique and that every X has one except $X^* = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. Switching $\min A$ and $\min B$ in X (if $X = X^*$, we do nothing) produces the matching X' (see Figure 1). It is clear that A and B remain the crucial pair of X' and that $s(X) - s(X') = \pm 1$ because the sets of crossing pairs of X and of X' differ exactly in the pair A, B. So $\Phi : X \mapsto X'$ is an involution that changes the parity of $s(X)$. It pairs even and odd matchings with the exception of X^* and $s(X^*) = 0$ is even. $\square$

A remarkable formula for the generating polynomial counting matchings by crossings was derived by Touchard and Riordan [16], [10] and was later proved purely combinatorially (using bijections between words, trees, and polyominoes) by Penaud [9]:

$$\sum_{X \in \mathcal{S}_n} x^{s(X)} = \frac{1}{(1-x)^n} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k} x^{k(k-1)/2}.$$

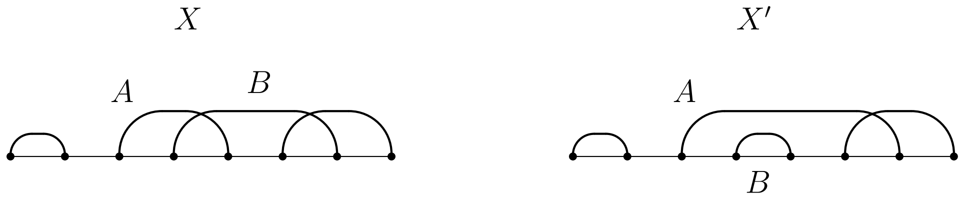

Figure 1: The involution Φ.

The reader is invited to do an exercise: recover the formulas for S_n and $S_n^{\pm}$ from the polynomial by setting $x = 1$ and $x = -1$.

6. CONCLUDING REMARKS. Theorem 2 follows from equation (1), which is proved in [11, p. 373]. Our derivation is more condensed. The analytic argument establishing Theorem 3 seems to be new. The same is perhaps true of the involution proof of Theorem 4, but the result itself, that $S_n^{\pm} = 1$, was found by Riordan [10, p. 219]. We conclude by stating a problem on *connected* matchings. These are matchings X with the following property: for each pair of distinct edges A and B of X there is a chain of edges $A_0, A_1, \ldots, A_k$ of X such that $A_0 = A$, $A_k = B$, and A_i and A_{i+1} are a crossing pair for $i = 0, 1, \ldots, k-1$. For example, both X and X' in Figure 1 are disconnected, having two and three components, respectively. Let $\mathcal{S}_n$ be the set of all connected matchings on $[2n]$, and let $s(X)$ again be the number of crossings. It is known and not too difficult to prove (see the articles of Stein [14] and Nijenhuis and Wilf [8]) that the sequence $(S_n)_{n\geq 1} = (1, 1, 4, 27, 248, 2830, \ldots)$ (this is A000699 of [12]) satisfies the recurrence relation $S_n = (n-1)\sum_{i=1}^{n-1} S_i S_{n-i}$. (For further results on matchings and crossings see Flajolet and Noy [4].) Now, as for $S_n^{\pm}$, do we have nice cancellation in the style of Theorems 1, 2, and 4, or do we have rather erratic behavior as in Theorem 3?

ACKNOWLEDGMENTS. I was supported by the project LN00A056 of the Ministry of Education of the Czech Republic. I am grateful to M. Noy for his comments. I would like to thank an anonymous referee for interesting and useful comments.

References

[1] G. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.

[2] N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover, New York, 1981.

[3] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999) 167–202.

[4] P. Flajolet and M. Noy, Analytic combinatorics of chord diagrams, in *Proceedings of FPSAC'00, Moscow 2000*, D. Krob, A. A. Mikhalev and A. V. Mikhalev, eds., Springer-Verlag, Berlin, 2000, pp. 191–201.

[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, UK, 1979.

[6] G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Math.* **1** (1972) 333–350.

[7] L. Lovász, *Combinatorial Problems and Exercises*, Akadémiai Kiadó, Budapest, 1993.

[8] A. Nijenhuis and H. S. Wilf, The enumeration of connected graphs and linked diagrams, *J. Comb. Theory, Ser. A* **27** (1979) 356–359.

[9] J.-G. Penaud, Une preuve bijective d'une formule de Touchard–Riordan, *Discrete Math.* **139** (1995) 347–360.

[10] J. Riordan, The distribution of crossings of chords joining pairs of $2n$ points on a circle, *Math. of Computation* **29** (1975) 215–222.

[11] R. Simion, Noncrossing partitions, *Discrete Math.* **217** (2000) 367–409.

[12] N. J. A. Sloane (2001), The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

[13] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Wadsworth & Brooks/Cole, Monterey, CA, 1986.

[14] P. R. Stein, On a class of linked diagrams, I. Enumeration, *J. Comb. Theory, Ser. A* **24** (1978) 357–366.

[15] M. V. Subbarao and A. Verma, Some remarks on a product expansion. An unexplored partition function, in *Symbolic Computations, Number Theory, Special Functions, Physics and Combinatorics*, F. G. Garvan and M. E. H. Ismail, eds., Kluwer, Dordrecht, 2001, pp. 267–283.

[16] J. Touchard, Sur un problème de configurations et sur les fractions continues, *Canad. J. Math.* **4** (1952) 2–25.

[17] Y. Yang, On a multiplicative partition function, *Electr. J. of Comb.* **8** (2001) R 19, 14 pages.

Department of Applied Mathematics (KAM), Charles University, and Institute for Theoretical Computer Science (ITI), Malostranské náměstí 25, 118 00 Praha, Czech Republic
klazar@kam.mff.cuni.cz