How many ordered factorizations may n have?

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Overview

1. Previous bounds on the number $m(n)$ of ordered factorizations of n

2. Our bounds

3. Outline of the proof

4. Further properties of $m(n)$

1. Previous bounds. It is well known that

$$c(n) = \#(\text{solutions to } n = \sum a_i,\ a_i \geq 1) = 2^{n-1}$$

but what is

$$m(n) = \#(\text{solutions to } n = \prod a_i,\ a_i \geq 2)\ ?$$

(The order of summands and factors matters.)

The values of $m(n)$ for $n = 1, 2, \ldots, 50$ are:

$$\begin{aligned}&0(1), 1, 1, 2, 1, 3, 1, 4, 2, 3, 1, 8, 1, 3, 3, 8, 1, 8,\\ &1, 8, 3, 3, 1, 20, 2, 3, 4, 8, 1, 13, 1, 16, 3, 3, 3, 26,\\ &1, 3, 3, 20, 1, 13, 1, 8, 8, 3, 1, \underline{48}, 2, 8, \ldots\end{aligned}$$

We have $m(n) < n$ for $n < 48$ but $m(48) = 48$. Perhaps $m(n) < n^{1.5}$ for all n?

Well, no, but it is true that $m(n) < n^\rho$ for all n when $\rho = 1.72864\ldots$, where ρ satisfies $\zeta(\rho) = \sum 1/n^\rho = 2$.

Proof by induction (Coppersmith & Lewenstein, 2005). $m(1) = 1 \le 1^\rho$ and for $n > 1$,

$$\begin{aligned} m(n) &= \sum_{d|n,\, d>1} m(n/d) \le \sum_{d|n,\, d>1} n^\rho/d^\rho \\ &< n^\rho \sum_{d>1} 1/d^\rho = n^\rho(\zeta(\rho) - 1) \\ &= n^\rho. \qquad \square \end{aligned}$$

Where does ρ come from? From the Dirichlet series

$$\sum_{n\ge1} \frac{m(n)}{n^s} = \sum_{r\ge0} (\zeta(s) - 1)^r = \frac{1}{2 - \zeta(s)}.$$

How big may $m(n)$ be? In average,

$$\sum_{n \le x} m(n) = (c + o(1))x^\rho, \ x \to \infty,$$

where $c = -1/\rho\zeta'(\rho) = 0.31817\ldots$ (Kalmár, 1931). Bounds on the error term:

$$\ll \exp(-\alpha_\varepsilon(\log\log x)^{4/3-\varepsilon}), \ \alpha_\varepsilon > 0$$

(Ikehara, 1941) and $4/3 \to 3/2$ by Hwang in 2000.

What about the maximal order of $m(n)$? Erdős claimed in 1941 that for some constants $0 < c_2 < c_1 < 1$,

$$\begin{array}{lcll} m(n) & < & n^\rho/\exp((\log n)^{c_2}) & \text{for } n > n_0 \\ m(n) & > & n^\rho/\exp((\log n)^{c_1}) & \text{for } \infty \text{ many } n \end{array}$$

but gave no proof. Best bounds proved so far are $m(n) < n^\rho$ (Chor, Lemke and Mador, 2000) and $m(n) > n^{\rho-\varepsilon}$ for ∞ many n (Hille, 1937).

2. Our bounds. One may take $c_1 = c_2 = 1/\rho$. More precisely, $\forall \varepsilon > 0$

$$m(n) < \frac{n^\rho}{\exp\left((\log n)^{1/\rho}/(\log\log n)^{1+\varepsilon}\right)}$$

holds for $n > n_0$, while, for some positive constant c,

$$m(n) > \frac{n^\rho}{\exp\left(c(\log n)^{1/\rho}/(\log\log n)^{1/\rho}\right)}$$

holds for ∞ many n.

(Klazar & Luca, arXiv:math.NT/0505352, version 2)

Perhaps $1 + \varepsilon \to 1/\rho$?

3. Outline of the proof. Let $p_k =$ the kth prime, $P(n) = \max_{p|n} p$,

$$\mathcal{P}_k = \{n : \ P(n) \le p_k\}$$

and

$$m_k(n) = \#(\text{solutions to } n = \prod a_i, \ a_i \in \mathcal{P}_k \backslash \{1\}).$$

So $m_k(n) = m(n)$ if $n \in \mathcal{P}_k$ and $m_k(n) = 0$ else. As before we have

$$m_k(n) < n^{\rho_k}$$

where $\zeta_k(\rho_k) = 2$ and $\zeta_k(s)$ is defined by

$$\zeta_k(s) = \prod_{p \le p_k} \left(1 - 1/p^s\right)^{-1} = \sum_{n \in \mathcal{P}_k} 1/n^s.$$

Clearly, $\rho_k \uparrow \rho$ as $k \to \infty$ but how fast?

$$\rho - \rho_k = \frac{c + O(\log\log k / \log k)}{k^{\rho-1}(\log k)^{\rho}}.$$

The lower bound. We use the Dirichlet series

$$\sum_{n\geq 1} \frac{m_k(n)}{n^s} = \sum_{r\geq 0} (\zeta_k(s)-1)^r = \frac{1}{2-\zeta_k(s)}.$$

By the effective Ikehara–Ingham theorem (due to Tenenbaum), for $x \to \infty$,

$$\sum_{n\leq x} m_k(n) = \sum_{n\leq x,\ P(n)\leq p_k} m(n) = (c_k + o(1))x^{\rho_k},$$

uniformly in $k = 2, 3, 4, \ldots$. Thus, with the usual notation $\Psi(x,y) = \#\{n \leq x : P(n) \leq y\}$, there exists an $N \leq x$ such that

$$m(N) > \frac{x^{\rho_k}}{5\Psi(x,p_k)} = \frac{x^{\rho}/\exp((\rho-\rho_k)\log x)}{5\Psi(x,p_k)}.$$

Using the asymptotics $\rho - \rho_k = \cdots$, bounds on $\Psi(x,y)$ and tunning $k = k(x)$, we obtain the lower bound.

The upper bound. We are given an $n = q_1^{a_1} \dots q_k^{a_k}$ where $2 \le q_1 < q_2 < \dots$. Let $\overline{n} = p_1^{a_1} \dots p_k^{a_k}$ (now $p_1 = 2$, $p_2 = 3, \dots$). So $\overline{n} \le n$ and

$$m(n) = m(\overline{n}) < \overline{n}^{\rho_k} \le n^{\rho_k}$$

where $k = \omega(n)$. If $k = \omega(n)$ is small, we bound

$$m(n) < n^{\rho_k} = \frac{n^{\rho}}{\exp((\rho - \rho_k) \log n)}$$

again by the asymptotics $\rho - \rho_k = \cdots$. But what if $k = \omega(n)$ is not small? By a combinatorial argument, if $q|n$ then

$$m(n) < 2\Omega(n) \cdot m(n/q) < 3 \log n \cdot m(n/q).$$

Iterating, we get

$$m(n) < (3 \log n)^k \cdot m(n/q_1 q_2 \dots q_k)$$

where $k = \omega(n)$.

So

$$\begin{aligned} m(n) &< (3\log n)^k \cdot m(n/q_1q_2\dots q_k) \\ &< (3\log n)^k \frac{n^\rho}{(q_1q_2\dots q_k)^\rho} \\ &\le (3\log n)^k \frac{n^\rho}{(p_1p_2\dots p_k)^\rho} \\ &= \frac{n^\rho}{\exp(\rho\sum_{i\le k}\log p_i - k(\log\log n + \log 3))}. \end{aligned}$$

If $k = \omega(n)$ is not small, $\exp(\cdots) > 1$ and we get a non-trivial upper bound. Tunning $k = k(n)$ and combining both arguments, we get the upper bound.

4. Further properties of $m(n)$. (1, 2, and 3 are proved in our preprint, 1 was obtained also by Knopfmacher and Mays in 2005).

1. $m(n) = n$ for ∞ many n.

2. $m(n)$ is odd $\iff$ n is squarefree.

3. Sequence $(m(n))_{n\geq 1}$ is not holonomic, that is, satisfies no linear recurrence with polynomial coefficients.

4. (Cayley, 1859; often rediscovered). If $a_k = m(p_1 p_2 \ldots p_k)$ then

$$\sum_{k\geq 0} \frac{a_k x^k}{k!} = \frac{1}{2 - \exp(x)}.$$

5. (MacMahon, 1893). A formula for $m(q_1^{a_1} \dots q_k^{a_k})$ in terms of the multiset $\{a_1, a_2, \dots, a_k\}$.

6. (MacMahon, 1893). $m(n)$ is equal to the number of perfect partitions of $n-1$.

$\lambda \vdash n$ is perfect if for every $n' \leq n$ there is exactly one subpartition λ' of λ with $\lambda' \vdash n'$.

Example. $m(12) = 8$ and 11 has 8 perfect partitions, namely $(1^2, 3, 6)$, $(1, 2^2, 6)$, $(1^5, 6)$, $(1, 2, 4^2)$, $(1^3, 4^2)$, $(1^2, 3^3)$, $(1, 2^5)$, and (1^{11}).