MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2023/24
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LECTURE 4 (March 12, 2024) PROOF OF FTAlg. COMPLETE SPACES. BAIRE'S THEOREM

• *n-th complex roots.* First, we realize that when proving the existence of n-th roots of complex numbers, it suffices to restrict to odd n and to numbers with modulus 1, i.e., lying on the complex unit circle S.

Exercise 1 *Using the last two exercises given in the previous lecture, prove that if for every $u \in S$ and for every odd $n \in \mathbb{N}$ there exists a $v \in S$ such that $v^n = u$, then the following theorem holds.*

Theorem 2 (n-th roots in $\mathbb{C}$) *Complex numbers contain all n-th roots, that is*

$$\forall\, u \in \mathbb{C}\; \forall\, n \in \mathbb{N}\; \exists\, v \in \mathbb{C}\, \big(v^n = u\big)\ .$$

Proof. By the previous exercise, we can assume that $u \in S$ and that $n \in \mathbb{N}$ is odd. We need to prove that the map

$$f(z) = z^n \colon S \to S\ ,$$

which is clearly continuous, is onto. We assume for contradiction that there is a number

$$w \in S \setminus f[S]$$

(i.e., w has no n-th root). Since n is odd, also $-w \in S \setminus f[S]$ (always $f(-z) = -f(z)$). We consider the line $\ell \subset \mathbb{C}$ going through w

and $-w$. Then we have the partition

$$\mathbb{C} = A \cup \ell \cup B\ ,$$

where A and B are open half-planes determined by the line ℓ. By Exercise 3 below, A and B are disjoint open sets. By Exercise 4 below, $(A \cup B) \cap S = S \setminus \{w, -w\}$, $\{1, -1\} \subset f[S] \cap (A \cup B)$ and $|A \cap \{1, -1\}| = 1$. Thus, the sets A and B cut the set $f[S]$ and it is disconnected. This contradicts Theorem 21 in the last lecture, because $f[S]$ is the image of the connected set S (we proved its connectedness last time) by the continuous function f and is therefore connected. □

Exercise 3 *Prove that for every line $\ell \subset \mathbb{C}$, $\mathbb{C}\setminus\ell$ is the disjoint union of two open sets.*

Exercise 4 *Let $\ell \subset \mathbb{C}$ be a line passing through the origin, $\ell \cap S = \{w, -w\}$ and A and B are the open half-planes determined by it. Prove that $(A \cup B) \cap S = S \setminus \{w, -w\}$ and that for every $u \in S\setminus\{w, -w\}$, the points u and $-u$ lie in different half-planes A and B.*

We proceed to the second step of the proof of FTAlg. Using compact subsets in $\mathbb{C}$, we deduce the FTAlg from the existence of n-th roots. Recall that the complex numbers $\mathbb{C}$ are the MS $(\mathbb{C}, |u-v|)$ which is isometric to the Euclidean space $(\mathbb{R}^2, e_2)$.

Exercise 5 *Prove that for every real numbers $\alpha \le \alpha'$ and $\beta \le \beta'$, the rectangle*

$$R = \{a + bi \mid \alpha \le a \le \alpha' \wedge \beta \le b \le \beta'\}$$

is a compact set.

Proposition 6 (reduction to n-th roots) *If $\mathbb{C}$ contains all n-th roots, then* FTAlg *holds – every non-constant complex polynomial has a root.*

Proof. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

be a non-constant complex polynomial, that is, $n \in \mathbb{N}$, $a_j \in \mathbb{C}$ and $a_n \neq 0$. The function

$$f(z) = |p(z)| \colon \mathbb{C} \to [0, +\infty) \subset \mathbb{C}$$

is continuous. We prove that $f(u) = 0$ for some $u \in \mathbb{C}$. Then also $p(u) = 0$ and u is a root of the polynomial $p(z)$.

First we prove that f attains on its definition domain $\mathbb{C}$ a minimum value $f(u)$, and then that $f(u) = 0$. Let the real number $K > 0$ be so large that

$$\frac{K^n |a_n|}{2} > |a_0| \text{ and } \sum_{j=0}^{n-1} |a_j| K^{j-n} < \frac{|a_n|}{2}\ .$$

Then for $z \in \mathbb{C}$ we have the estimate that

$$\begin{aligned} |z| > K \Rightarrow f(z) = |p(z)| \ &\geq \ |z|^n \left(|a_n| - \sum_{j=0}^{n-1} |a_j| \cdot |z|^{j-n} \right) \\ &> \ |a_0| = |p(0)| = f(0)\ . \end{aligned}$$

We define a rectangle

$$R = \{a + bi \mid -K \leq a,\, b \leq K\} \subset \mathbb{C}\ .$$

Clearly, $z \in \mathbb{C} \backslash R \Rightarrow |z| > K$. By Theorem 15 in the second lecture (maximum principle) and Exercise 5 there exists $u \in R$ such that

$f(u) \le f(v)$ for every $v \in R$. Since $0 \in R$, $f(u) \le f(0)$. By the above estimate we have that

$$\forall\, v \in \mathbb{C} \left(f(u) \le f(v) \right) .$$

Thus f attains at u its smallest value on the whole $\mathbb{C}$.

We prove that $f(u) = 0$. For this purpose we express the polynomial $p(z)$ by Exercise 7 in the form

$$p(z) = \sum_{j=0}^{n} b_j (z-u)^j \ ,$$

where $b_j \in \mathbb{C}$ and $b_n = a_n$. Thus, in this expression, $f(u) = |p(u)| = |b_0|$. We assume for contrary that $f(u) = |b_0| > 0$. We find the first non-zero non-constant coefficient in the polynomial $p(z)$ and write $p(z)$ as

$$p(z) = b_0 + b_k(z-u)^k + \underbrace{b_{k+1}(z-u)^{k+1} + \cdots + b_n(z-u)^n}_{q(z)} ,$$

where $q \in \mathbb{C}[z]$, $k \in \mathbb{N}$, $b_0 \ne 0$ and $b_k \ne 0$. We use the assumption about n-th roots and take an $\alpha \in \mathbb{C}$ such that

$$\alpha^k = -\frac{b_0}{b_k} \ .$$

It is clear that $q(z) = o((z-u)^k)$ (for $z \to u$), so that

$$\lim_{z \to u} q(z)(z-u)^{-k} = 0 \ .$$

So we can take a $\delta \in (0,1)$ such that for

$$v = u + \delta\alpha$$

one has that

$$|q(v)| < \delta^k \cdot \frac{|b_0|}{2} \ .$$

Then we get the contradiction that $f(v) < f(u)$:

$$\begin{aligned} f(v) = |p(v)| &= |b_0 + b_k\alpha^k\delta^k + q(v)| \\ &\overset{\text{def. of } \alpha}{=} |b_0(1-\delta^k) + q(v)| \\ &\overset{\Delta\text{'s ineq. and mult. } |\cdot|}{\leq} |b_0|(1-\delta^k) + |q(v)| \\ &\overset{|q(v)| < \dots}{<} |b_0|(1-\delta^k/2) \\ &\overset{\delta \in (0,\,1)}{<} |b_0| = f(u) \ . \end{aligned}$$

So $f(u) = 0$ and $p(u) = 0$. □

Exercise 7 *Prove that for every* $n \in \mathbb{N}_0$ *and any complex numbers* $a_0,\ a_1,\ \dots,\ a_n$ *and* u *there exist complex numbers* $b_0,\ b_1,\ \dots,\ b_n$ *such that* $b_n = a_n$ *and the polynomial equality*

$$\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} b_j (z-u)^j$$

holds.

• *Complete sets and complete* MS*s.* A MS (M, d) is *complete* if every Cauchy sequence $(a_n) \subset M$ is convergent. A *Cauchy sequence* (a_n) is one such that

$$\forall\, \varepsilon\ \exists\, n_0 \left(m,\, n \geq n_0 \Rightarrow d(a_m,\, a_n) < \varepsilon\right) .$$

A set $X \subset M$ is *complete* if the subspace (X, d) is complete.

Exercise 8 *Let* (M, d) *be* MS *and* $X \subset Y \subset M$*. Prove that a set* X *is complete in the* MS (Y, d) *if and only if it is complete in the* MS (M, d)*.*

Exercise 9 *Prove that the Cartesian product*

$$(M \times N,\, d \times e)$$

of complete MS*s* (M, d) *and* (N, e) *is a complete* MS*.*

A basic example of a complete MS is the Euclidean space

$$(\mathbb{R},\, e_1) = (\mathbb{R},\, |x - y|)\ ,$$

which is complete due to the fact that every sequence $(a_n) \subset \mathbb{R}$ is convergent if and only if it is Cauchy. By Exercise 9 all Euclidean spaces $(\mathbb{R}^n, e_n)$, $n \in \mathbb{N}$, are complete. We can construct many complete MSs by means of the following simple result.

Proposition 10 (closed subspaces) *In every complete* MS (M, d) *every closed subset* $X \subset M$ *is complete.*

Proof. Let $(a_n) \subset X$ be a Cauchy sequence in the closed set $X \subset M$ in the complete MS (M, d). There exists $a = \lim a_n \in M$. Since X is a closed set, $a \in X$ (closed sets are closed also to limits). So the set X is complete. □

Exercise 11 *Let* $X \subset M$ *be a compact set in a* MS (M, d)*. Prove that* X *is complete.*

Exercise 12 *Give an example of a complete and non-compact set* $X \subset \mathbb{R}$ *in the Euclidean* MS $(\mathbb{R}, e_1)$*.*

Exercise 13 *Which of the following implications holds in a* MS (M, d)*?*

1. $X \subset M$ *is a complete set* $\Rightarrow X$ *is closed.*
2. $X \subset M$ *and* $Y \subset M$ *are complete sets* $\Rightarrow X \cup Y$ *is a complete set.*
3. $X \subset M$ *and* $Y \subset M$ *are complete sets* $\Rightarrow X \cap Y$ *is a complete set.*
4. $X \subset M$ *is a complete set* $\Rightarrow X$ *is bounded.*
5. $X \subset M$ *is finite* $\Rightarrow X$ *is complete.*

• *Baire's theorem.* The main result about complete MSs is, besides completeness of particular MSs, Baire's theorem: no complete MS is a countable union of sparse sets. A set $X \subset M$ in a MS (M, d) is *sparse (in M)* if

$$\forall\, a \in M\ \forall\, r > 0\ \exists\, b \in M\ \exists\, s > 0$$
$$\big(B(b,\, s) \subset B(a,\, r) \wedge B(b,\, s) \cap X = \emptyset\big)\ .$$

In words, every ball in the MS (M, d) contains a subball disjoint to X.

Similarly, a set $X \subset M$ in a MS (M, d) is *dense (in M)* if

$$\forall\, a \in M\ \forall\, r > 0\, \big(B(a,\, r) \cap X \neq \emptyset\big)\ .$$

In words, every ball in the MS (M, d) contains an element of the set X.

Exercise 14 *Let* (M, d) *be a* MS *and* $X \subset M$ *be a subset. Prove the equivalence that*

$$X \textit{ is dense} \iff \forall\, a \in M\ \exists\, (a_n) \subset X\ \big(\lim a_n = a\big)\ .$$

Proposition 15 (density and continuity) *Let* (M,d) *and* (N,e) *be* MS*s,* $X \subset M$ *be dense in* M *and let*

$$f,\, g\colon M \to N$$

be continuous mappings such that $f\,|\,X = g\,|\,X$ *(their restrictions to the set* X *coincide). Then* $f = g$*.*

Proof. Let $a \in M$ be an arbitrary point. Since X is dense, by the previous exercise there exists a sequence $(a_n) \subset X$ such that $\lim a_n = a$. Using Heine's definition of continuity of functions and the assumption about f and g, we have that

$$f(a) = \lim\, f(a_n) = \lim\, g(a_n) = g(a).$$

So $f = g$. □

Exercise 16 *Any finite union of sparse sets is a sparse set. Show by an example that this is not generally true for countable unions.*

Exercise 17 *Prove that the intersection of two dense sets, one of which is open, is a dense set. Show that this is not in general true if we omit the assumption of openness.*

For $a \in M$ and real $r > 0$, the *closed ball* $\overline{B}(a,r)$ in a MS (M,d) is the set

$$\overline{B}(a,\, r) = \{x \in M \mid d(a,\, x) \le r\}\ .$$

Exercise 18 *Every closed ball* $\overline{B}(a,r)$ *is a closed set. For every* $a \in M$ *and* $r,s \in \mathbb{R}$ *with* $0 < r < s$*,*

$$\overline{B}(a,\, r) \subset B(a,\, s)\ .$$

Theorem 19 (Baire's) *Let* (M, d) *be a complete* MS *and*

$$M = \bigcup_{n=1}^{\infty} X_n \ .$$

Then for some n*, the set* X_n *is not sparse. In other words, no complete metric space is a countable union of sparse sets.*

Proof. We assume that all sets X_n are sparse and deduce a contradiction. We construct a nested sequence $(\overline{B_n})$ of closed balls with centers converging to a point $a \in M$ outside any X_n, which is clearly a contradiction.

Let $B(b, 1) \subset M$ be an arbitrary ball. Since X_1 is sparse, there exists an $a_1 \in M$ and an $s_1 > 0$ such that $B(a_1, s_1) \subset B(b, 1)$ and $B(a_1, s_1) \cap X_1 = \emptyset$. We set

$$\overline{B}(a_1,\, r_1) = \overline{B}(a_1,\, \min(s_1/2,\, 1/2)) \ .$$

Then $\overline{B}(a_1, r_1) \subset B(a_1, s_1)$, thus $\overline{B}(a_1, r_1) \cap X_1 = \emptyset$, and $r_1 \leq 1/2$.

Suppose that we already defined the closed balls

$$\overline{B}(a_1,\, r_1) \supset \overline{B}(a_2,\, r_2) \supset \cdots \supset \overline{B}(a_n,\, r_n)$$

such that for $i = 1, 2, \ldots, n$, $\overline{B}(a_i, r_i) \cap X_i = \emptyset$ and $r_i \leq 2^{-i}$. Since X_{n+1} is sparse, there exist $a_{n+1} \in M$ and $s_{n+1} > 0$ such that $B(a_{n+1}, s_{n+1}) \subset B(a_n, r_n)$ and $B(a_{n+1}, s_{n+1}) \cap X_{n+1} = \emptyset$. We set

$$\overline{B}(a_{n+1},\, r_{n+1}) = \overline{B}(a_{n+1},\, \min(s_{n+1}/2,\, 2^{-n-1})) \ .$$

Then

$$\overline{B}(a_{n+1},\, r_{n+1}) \subset \overline{B}(a_n,\, r_n) \cap B(a_{n+1},\, s_{n+1}) \ ,$$

hence also $\overline{B}(a_{n+1}, s_{n+1}) \cap X_{n+1} = \emptyset$, and $r_{n+1} \leq 2^{-n-1}$.

The sequence $(a_n) \subset M$ of the centers of the closed balls defined above is Cauchy, since

$$m \geq n \Rightarrow \overline{B}(a_m,\, r_m) \subset \overline{B}(a_n,\, r_n), \ \text{ so } \ d(a_m,\, a_n) \leq r_n \leq \frac{1}{2^n}\ .$$

We use completeness of the MS (M, d) and take the limit

$$a = \lim a_n \in M\ .$$

Since $m \geq n \Rightarrow a_m \in \overline{B}(a_n, r_n)$ and since by Exercise 18 every $\overline{B}(a_n, r_n)$ is a closed set, the limit a lies in every closed ball $\overline{B}(a_n, r_n)$ and therefore in none of the sets X_n, which is a contradiction. $\square$

Baire's theorem has many applications, of which we now mention only one. A point $a \in M$ in a MS (M, d) is *isolated* if

$$\exists\, r > 0\, \big(B(a,\, r) = \{a\}\big)\ .$$

Exercise 20 *Prove that in any* MS (M, d)*,*

$a \in M$ *is not an isolated point* $\iff$ $\{a\} \subset M$ *is a sparse set .*

Corollary 21 (getting uncountability) *Any complete* MS (M, d) *without isolated points is uncountable.*

Proof. Suppose for the contrary that M is countable. Then

$$M = \bigcup_{a \in M} \{a\}$$

is a countable union. Since each set $\{a\}$ is sparse (by the previous exercise), we have a contradiction with Baire's theorem. $\square$

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 7, 11, 13, 16 and 20.