MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2023/24
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LECTURE 11 (April 30, 2024) INTRODUCTION TO COMPLEX ANALYSIS 3

• *A theorem on the integral* $\int_{\partial R}$. In an analogy to the last theorem of the previous lecture we obtain basic properties of the integral $\int_{\partial R} f$ for holomorphic functions $f\colon \mathbb{C}\setminus A\to\mathbb{C}$, where $A\subset \mathrm{int}(R)$ is a compact set and $R\subset\mathbb{C}$ is a rectangle.

Theorem 1 (properties of $\int_{\partial R}$) *These $\int$s have three important properties.*

1. *(linearity) If R, A and functions f and g are as above, then for every $\alpha,\beta\in\mathbb{C}$,*
$$\int_{\partial R}(\alpha f+\beta g)=\alpha\int_{\partial R} f+\beta\int_{\partial R} g\ .$$
2. *(An extension of the C.–G. thm.) If R, $A=\{a\}\subset\mathbb{C}$ and f are as above and f is bounded on a deleted neighborhood of the point a, then*
$$\int_{\partial R} f=0\ .$$
3. *For every $a\in\mathbb{C}$ and every rectangle $R\subset\mathbb{R}$ with $a\in\mathrm{int}(R)$,*
$$\int_{\partial R}\frac{1}{z-a}=\rho$$
where $\rho=2\pi i$ is the previously introduced constant.

Proof. 1. We actually proved this linearity earlier, in the last theorem of the lecture before the last lecture.

2. We take some rectangles R_n containing the point a in their interiors and shrink them to a. ML estimates of the integrals $\int_{\partial R_n} f$ then show, due to $\mathrm{per}(R_n) \to 0$ as $n \to \infty$ and to boundedness of $|f|$ on a deleted neighborhood of the point a, that these integrals go to 0. Hence $\int_{\partial R} f = 0$.

3. Let S be a square with vertices $\pm 1 \pm i$ and $a + S$ be its shifted copy. By the previous theorem, by the definition of $\int_{\partial R}$ and by the definition of the constant ρ we have that

$$\int_{\partial R} \frac{1}{z-a} = \rho \int_{\partial(a+S)} \frac{1}{z-a} = \int_{\partial S} \frac{1}{z} = \rho \; .$$

□

Exercise 2 *Let R be a rectangle, $a \in \mathrm{int}(R)$ be a point, and $k \geq 2$ be an integer. Then*

$$\int_{\partial R} \frac{1}{(z-a)^k} = 0 \; .$$

• *The Cauchy formula.* For simplicity we state and prove it only for entire functions.

Theorem 3 (Cauchy formula) *Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function, $\rho = 2\pi i$ be the previously defined constant, $R \subset \mathbb{C}$ be a rectangle and $a \in \mathrm{int}(R)$. Then*

$$f(a) = \frac{1}{\rho} \int_{\partial R} \frac{f(z)}{z-a} \; .$$

Proof. The existence of the derivative $f'(a)$ implies the boundedness of the function

$$\frac{f(z) - f(a)}{z-a}$$

on a deleted neighborhood of the point a. By 1–3 of Theorem 1 we have that

$$0 \stackrel{\text{part 2}}{=} \int_{\partial R} \frac{f(z)-f(a)}{z-a} \stackrel{\text{part 1}}{=} \int_{\partial R} \frac{f(z)}{z-a} - f(a)\int_{\partial R} \frac{1}{z-a}$$
$$\stackrel{\text{part 3}}{=} \int_{\partial R} \frac{f(z)}{z-a} - f(a)\rho\ .$$

Since $\rho \neq 0$ (Theorem 6 in the last lecture), we immediately get the Cauchy formula. □

Proof of Liouville's theorem. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire bounded function, so $|f(z)| < c$ for every $z \in \mathbb{C}$ and some real constant $c > 0$. Let $a, b \in \mathbb{C}$ be two (different) points. Using Exercise 4, for every sufficiently large $s \in \mathbb{N}$ we find a square $S \subset \mathbb{C}$ with side of length s such that $a, b \in \mathrm{int}(S)$ and for every $z \in \partial S$,

$$|z-a|,\ |z-b| > \frac{s}{3} = \frac{\mathrm{per}(S)}{12}\ .$$

Due to the Cauchy formula and the linearity of $\int_{\partial R}$,

$$\begin{aligned} f(a)-f(b) &= \frac{1}{\rho}\int_{\partial S} \frac{f(z)}{z-a} - \frac{1}{\rho}\int_{\partial S} \frac{f(z)}{z-b} \\ &= \frac{a-b}{\rho}\int_{\partial S} \frac{f(z)}{(z-a)(z-b)}\ . \end{aligned}$$

By the ML estimate, the last integral is in the absolute value at most

$$\frac{c}{\mathrm{per}(S)^2/144} \cdot \mathrm{per}(S) = \frac{144c}{4s} = \frac{36c}{s} \to 0 \ \text{ for } \ s \to \infty\ .$$

So $f(a) = f(b)$ and f is a constant function. □

Exercise 4 *Let $a, b \in \mathbb{C}$. For every large $s \in \mathbb{N}$, find a square $S \subset \mathbb{C}$ with side length s such that $a, b \in \mathrm{int}(S)$ and for any $z \in \partial S$ the distances $|z-a|, |z-b|$ are greater than $s/3$.*

Proof of analyticity of every entire function. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, the number $a \in \mathbb{C}$ be arbitrary, and R be a large rectangle such that $0, a \in \mathrm{int}(R)$ and for every $z \in \partial R$,

$$\left|\frac{a}{z}\right| = \frac{|a|}{|z|} < \frac{1}{2} \ \text{ and } \ |z-a| > 1$$

(Exercise 5). Let $m \in \mathbb{N}$. Using Cauchy's formula and the identity

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^m + \frac{x^{m+1}}{1-x}$$

we get that

$$\begin{aligned}
f(a) &\overset{\text{C. formula}}{=} \frac{1}{2\pi i}\int_{\partial R} \frac{f(z)}{z-a} \\
&\overset{\text{identity}}{=} \frac{1}{2\pi i}\int_{\partial R} \frac{f(z)}{z}\left(\sum_{n=0}^{m}(a/z)^n + \frac{(a/z)^{m+1}}{1-a/z}\right) \\
&\overset{\text{linearity of } \int_{\partial R}}{=} \sum_{n=0}^{m}\left(\frac{1}{2\pi i}\int_{\partial R}\frac{f(z)}{z^{n+1}}\right)a^n + \frac{1}{2\pi i}\int_{\partial R}\frac{f(z)(a/z)^{m+1}}{z-a} \\
&=: \sum_{n=0}^{m} c_n a^n + \frac{I_{m+1}}{2\pi i}\ .
\end{aligned}$$

The ML estimate of the integral I_{m+1} shows that we are done: for $m \to \infty$,

$$|I_{m+1}| \le \max_{z \in \partial R}|f(z)| \cdot \frac{(1/2)^{m+1}}{1} \cdot \mathrm{per}(R) \to 0\ .$$

Thus, for every $a \in \mathbb{C}$ one has that

$$f(a) = \sum_{n=0}^{\infty} c_n a^n, \quad \text{where} \quad c_n = \frac{1}{2\pi i} \int_{\partial S} \frac{f(z)}{z^{n+1}}$$

with an arbitrary rectangle S containing inside the point 0. $\square$

Exercise 5 *Show that for any $a \in \mathbb{C}$ there exists a rectangle R such that $0, a \in \mathrm{int}(R)$ and for every $z \in \partial R$ one has that $|a/z| < \frac{1}{2}$ and $|z - a| > 1$.*

• *Meromorphic functions and residues.* We generalize part 3 of Theorem 1 about the constant ρ. A set $A \subset \mathbb{C}$ is *discrete* if every open disc $B(z, r) \subset \mathbb{C}$ contains only finitely many of its elements. A holomorphic function

$$f \colon U \setminus A \to \mathbb{C} \,,$$

where $A \subset U$ is discrete, is *meromorphic* and A is the *set of its poles* if every point $a \in A$ has a neighborhood $U_a \subset U$ with $U_a \cap A = \{a\}$ such that for a holomorphic function $g_a \colon U_a \to \mathbb{C}$ and some numbers $k_a \in \mathbb{N}_0$ and $c_{j,a} \in \mathbb{C}$, $j = 1, 2, \ldots, k_a$, it holds for any $z \in U_a \setminus \{a\}$ that

$$f(z) = g_a(z) + \sum_{j=1}^{k_a} \frac{c_{j,a}}{(z-a)^j} \,.$$

For $k_a = 0$, the sum is defined as 0 and the function $f = g_a$ is then holomorphic on U_a. The coefficient $c_{1,a}$ is the so-called *residue of the function f at the point a* and we denote it as

$$\mathrm{res}(f, \, a) := c_{1,\,a} \,.$$

From Cauchy's formula it follows that $\mathrm{res}(f, a)$ is uniquely determined by the function f (Exercise 7).

Theorem 6 (on residues) *We assume that* $f\colon U\setminus A\to\mathbb{C}$ *is a meromorphic function with the set of poles* A *and that* $R\subset U$ *is a rectangle whose boundary* ∂R *contains no point of* A*. Then the equality holds*

$$\frac{1}{2\pi i}\int_{\partial R} f=\sum_{a\in A\,\cap\,\mathrm{int}(R)}\mathrm{res}(f,\,a)=\sum_{a\in A\,\cap\,R}\mathrm{res}(f,\,a)$$

(both sums are finite). So the integral of the function f *over the boundary of the rectangle* R*, divided by* $2\pi i$*, is equal to the sum of the residues of the function* f *in poles lying inside* R*.*

Proof. The infinity of the intersection of $A\cap R$ would mean the existence of a limit point of the set A, in contradiction with its discreteness (Exercise 8). The above sums are therefore finite. We take mutually disjoint squares

$$S_a\subset\mathrm{int}(R)\cap U_a,\ a\in R\cap A\ ,$$

where U_a is the neighborhood of the point a from the definition of a meromorphic function and S_a has its center at a. We then divide the rectangle R into rectangles including all of these squares S_a and get that

$$\begin{aligned}\int_{\partial R} f &= \sum_{a\in A\,\cap\,R}\int_{\partial S_a} f=\sum_{a\in A\,\cap\,R}\int_{\partial S_a}\left(g_a(z)+\sum_{j=1}^{k_a}\frac{c_{j,a}}{(z-a)^j}\right)\\ &= \sum_{a\in A\,\cap\,R} 2\pi i\cdot\mathrm{res}(f,\,a)\end{aligned}$$

by which we are done. The first equality follows using part 3 of the theorem on properties of $\int_u$ (Exercise 9). The second equality follows from the definition of a meromorphic function. The third equality follows from the linearity of the integral, the Cauchy–Goursat theorem, part 3 of Theorem 1 and from Exercise 2. □

Exercise 7 *Why is the reside of a function f at a point a uniquely determined by the function f?*

Exercise 8 *Prove that every infinite subset of any rectangle R has a limit point.*

Exercise 9 *Show how to partition a given rectangle R, with prescribed disjoint rectangles $R_1, R_2, \ldots, R_k$ contained in $\mathrm{int}(R)$, by appropriate lines into subrectangles including all R_j such that the first equality in the above proof holds.*

Exercise 10 *What is the punch line of the following mathematical joke?*

> *Did you know that the contour integral of f around the border of France is zero? ??? Because all Poles are in Eastern Europe!*

• *Solution of the generalized Basel problem.* We now illustrate the usefulness of the residue theorem and complex analysis by summing the series ($k \in \mathbb{N}$)

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \,.$$

In the earlier lectures, we used Fourier series to show that $\zeta(2) = \pi^2/6$. We now generalize this formula for $\zeta(2k)$. But first we prove two auxiliary results.

Proposition 11 (on $F(z)$) *Let*

$$F(z) := \frac{2\pi i}{\mathrm{e}^{2\pi i z} - 1} : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C} \,.$$

The function F is meromorphic with the poles $\mathbb{Z}$. At any integer it has the residue 1.

Proof. (Sketch.) The function $f(z) := \mathrm{e}^{2\pi i z} - 1$ is entire (it is defined by the sum of the power series) and $f(z) = 0 \iff z \in \mathbb{Z}$. For any $n \in \mathbb{Z}$ we have the local expansion

$$f(z) = 2\pi i(z-n) + a_2(z-n)^2 + \ldots$$

since $f'(n) = 2\pi i$. Therefore, in a deleted neighborhood of n one has that

$$\begin{aligned} F(z) &= \frac{2\pi i}{f(z)} = \frac{1}{z-n} \cdot \frac{1}{1 + (a_2/2\pi i)(z-n)) + \cdots} \\ &= (z-n)^{-1} + b_0 + b_1(z-n) + \cdots \end{aligned}$$

and the residue of the function $F(z)$ at n is equal to 1. $\square$

Lemma 12 *Let $F(z)$ be as above and $S_N \subset \mathbb{C}$, for $N \in \mathbb{N}$, be the square with the vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. Then there exists a $c > 0$ such that*

$$\forall\, N \in \mathbb{N}\; \forall\, z \in \partial S_N \left(|F(z)| \le c\right) .$$

Proof. Since $F(z) = 2\pi i/(\mathrm{e}^{2\pi i z} - 1)$, for the given z it suffices to cut off $\mathrm{e}^{2\pi i z} - 1$ from 0 . For $z \in \partial S_N$ with $|\mathrm{im}(z)| \ge 1$,

$$\begin{aligned} \left|\mathrm{e}^{2\pi i z} - 1\right| &\ge \left|\left|\mathrm{e}^{2\pi i z}\right| - 1\right| = \left|\mathrm{e}^{\mathrm{re}(2\pi i z)} - 1\right| \\ &= \left|\mathrm{e}^{-2\pi \cdot \mathrm{im}(z)} - 1\right| \ge \min\left(1 - \mathrm{e}^{-2\pi},\, e^{2\pi} - 1\right) \\ &= 1 - \mathrm{e}^{-2\pi} > 0 \,. \end{aligned}$$

For $z \in \partial S_N$ with $|\mathrm{im}(z)| \leq 1$ we use the fact that the function $\mathrm{e}^{2\pi i z}$ is 1-periodic, and therefore we can after reduction modulo 1 work just in the strip P given by the condition $0 \leq \mathrm{re}(z) \leq 1$. Then $z = \frac{1}{2} + ix$, where $x \in \mathbb{R}$ with $|x| \leq 1$, and

$$\left|\mathrm{e}^{2\pi i z} - 1\right| = \left|\mathrm{e}^{\pi i}\mathrm{e}^{-2\pi x} - 1\right| = \left|\mathrm{e}^{-2\pi x} + 1\right| \geq 1 + \mathrm{e}^{-2\pi} \ .$$

Thus, one can set $c = 2\pi/(1 - \mathrm{e}^{-2\pi})$. □

Theorem 13 (summing $\sum n^{-2k}$) *For every $k \in \mathbb{N}$ there exists a positive fraction $\alpha_k \in \mathbb{Q}$ that*

$$\zeta(2k) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \cdots = \alpha_k \pi^{2k} \ .$$

Proof. There exist fractions $B_0, B_1, \ldots$, so-called *Bernoulli numbers*, such that

$$\frac{x}{\mathrm{e}^x - 1} =: \sum_{r=0}^{\infty} \frac{B_r x^r}{r!} \in \mathbb{Q}[[x]]$$

(Exercise 14). We take the familiar meromorphic function

$$F(z) = \frac{2\pi i}{\mathrm{e}^{2\pi i z} - 1} : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C} \ .$$

By Proposition 11 it has poles exactly in $\mathbb{Z}$ and with residues $\mathrm{res}(F, n) = 1$ for every $n \in \mathbb{Z}$. If $f(z)$ is holomorphic on a neighborhood of $n \in \mathbb{Z}$ then clearly $\mathrm{res}(fF, n) = f(n)$ (Exercise 15). We put $f(z) := 1/z^{2k}$. For $N \in \mathbb{N}$ we denote by S_N the familiar square with the vertices $(N+\frac{1}{2})(\pm 1 \pm i)$. By the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial S_N}\frac{F(z)}{z^{2k}} &= \sum_{n=-N}^{N}\operatorname{res}(F(z)z^{-2k},\, n)\\ &= \operatorname{res}(F(z)z^{-2k},\, 0)+2\sum_{n=1}^{N}\frac{1}{n^{2k}}\ .\end{aligned}$$

By Lemma 12 there is a constant $c>0$ such that for every $N\in\mathbb{N}$, $z\in\partial S_N \Rightarrow |F(z)|\le c$. According to the ML estimate, the above integral is in absolute value at most

$$\max_{z\in\partial S_N}\left|\frac{F(z)}{z^{2k}}\right|\cdot \operatorname{per}(S_N)\le \frac{c}{N^{2k}}\cdot(8N+4)\to 0 \ \text{ for } \ N\to\infty\ .$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=-\frac{1}{2}\cdot\operatorname{res}(F(z)z^{-2k},\, 0)\ .$$

By the definitions of the function $F(z)$ and Bernoulli numbers, we have that

$$F(z)z^{-2k}=\frac{2\pi i z\cdot z^{-1-2k}}{\mathrm{e}^{2\pi i z}-1}=\sum_{r=0}^{\infty}\frac{B_r(2\pi i)^r z^{r-1-2k}}{r!}\ .$$

Therefore (we take $r=2k$)

$$\operatorname{res}(F(z)z^{-2k},\, 0)=\frac{(-1)^k B_{2k}(2\pi)^{2k}}{(2k)!}$$

and the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\underbrace{\frac{2^{2k-1}}{(2k)!}(-1)^{k+1}B_{2k}}_{\alpha_k}\cdot\pi^{2k}$$

is indeed a rational multiple of π^{2k}. □

Exercise 14 *Prove that Bernoulli numbers are fractions.*

Exercise 15 *Prove that if $f(z)$ is holomorphic on a neighborhood of $n \in \mathbb{Z}$, then $\mathrm{res}(fF, n) = f(n)$.*

For $k \geq 2$, $B_{2k-1} = 0$ (Exercise 16). Further, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and so on (Exercise 17). The above proof is taken from the book P. D. Lax and L. Zalcman, *Complex Proofs of Real Theorems*, AMS (The American Mathematical Society), Providence, RI (Rhodes Island), 2012. You can learn more about Complex Analysis in Czech in the book of J. Veselý, *Komplexní analýza pro učitele*, Karolinum, Praha, 2000.

Exercise 16 *Prove that the Bernoulli numbers with odd indices > 1 are zero.*

Exercise 17 *Check the values of the Bernoulli numbers above.*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 4, 5, 10 and 14.