Lecture 2. $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ again — the right GF proof. D-finite and algebraic FPS

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Last time we justified the equality

$$\big(\text{the } f \in \mathbb{R}[[x]] \text{ s. t. } f(x)^2 = 1-4x \text{ and } f(0)=1\big) = \textstyle\sum_{n=0}^{\infty}\binom{1/2}{n}(-4)^n x^n$$

which was needed to get the formula $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ by solving the quadratic equation $C^2 - C + x = 0$. Literature on enumerative combinatorics glosses over this justification. Now we give the "right" simple derivation of the formula by generating functions. The proper way to make use of the equation $C^2 - C + x = 0$ is not to apply the quadratic formula (!) but to differentiate it. Recall that $C = C(x) = \sum_{n\ge1} C_n x^n$ is the generating function of Catalan numbers.

The third proof of Theorem 1 ($C_n = \frac{1}{n}\binom{2n-2}{n-1}$) in Lecture 1. From $C^2 - C + x = 0$ we have that $-C^2 + C = x$. Differentiating $C^2 - C + x = 0$ by x we thus get

$$2CC' - C' + 1 = 0 \leadsto C' = \frac{1}{1-2C} = \frac{\frac{1}{2}C - \frac{1}{4}}{-C^2 + C - \frac{1}{4}} = \frac{2C-1}{4x-1}$$

and $(4x-1)C' - 2C + 1 = 0$. Since $C = \sum_{n\ge1} C_n x^n$ and $C' = \sum_{n\ge1} nC_n x^{n-1}$, equating for $n \in \mathbb{N}$ the coefficient of x^n on the left side to zero gives the equation

$$4nC_n - (n+1)C_{n+1} - 2C_n = 0\ .$$

Thus $C_1 = 1$ and $C_{n+1} = \frac{4n-2}{n+1}\cdot C_n$ for any $n \in \mathbb{N}$. Hence for $n \ge 2$,

$$\begin{aligned} C_n &= \prod_{k=2}^{n}\frac{2(2k-3)}{k} = \frac{2^{n-1}\cdot(2n-3)!!}{n!} \\ &= \frac{2^{n-1}(n-1)!\cdot(2n-3)!!}{(n-1)!\cdot n!} = \frac{1}{n}\binom{2n-2}{n-1}\ . \end{aligned}$$

□

We have a new simple recurrence for C_n: $C_n = 1$ and for $n \in \mathbb{N}$,

$$C_{n+1} = \frac{4n-2}{n+1}\cdot C_n\ .$$

As $0 < \frac{4n-2}{n+1} < 4$ for $n \in \mathbb{N}$, induction gives the exponential upper bound $C_n < 4^n$. The exact asymptotics of the Catalan numbers follows from the Stirling formula $n! = (1+o(1)) \cdot \sqrt{2\pi n} \cdot (n/e)^n$. We get that for some constant $c > 0$,

$$C_n = (c + o(1)) \cdot n^{-3/2} \cdot 4^n \quad (n \to \infty) \ .$$

In the previous proof we obtained the differential equation

$$(4x-1)C' - 2C + 1 = 0 \ .$$

It means that $C \in \mathbb{R}[[x]]$ is a D-*finite* FPS. We define this class of FPS.

Definition 1 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. We say that a* FPS $f(x) = \sum_{n \geq 0} a_n x^n$ *in $K[[x]]$ is* D-*finite (or holonomic) if there exist polynomials p_{-1}, p_0, ..., p_k in $K[x]$, $k \in \mathbb{N}_0$, such that $p_k \neq 0$ and*

$$\sum_{i=-1}^{k} p_i(x) f^{(i)}(x) = 0 \ .$$

Here $f^{(-1)}(x) := 1$, $f^{(0)}(x) := f(x)$ and $f^{(i)}(x)$ for $i \in \mathbb{N}$ is the i-th derivative of f,

$$f^{(i)}(x) = \sum_{n=0}^{\infty} a_n n(n-1) \dots (n-i+1) x^{n-i} \ .$$

In words, $f(x)$ satisfies a (non-homogeneous) linear differential equation with polynomial coefficients. As usual, for $i = 1, 2, 3$ we write for $f^{(i)}$ synonymously f', f'' and f'''. By repeated differentiation we homogenize the equation and get that $p_{-1} = 0$.

$C = C(x)$ satisfies also the equation

$$C^2 - C + x = 0$$

and so is an example of an *algebraic* FPS.

Definition 2 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. We say that a* FPS $f(x) = \sum_{n \geq 0} a_n x^n$ *in $K[[x]]$ is algebraic if there exist polynomials p_0, ..., p_k in $K[x]$, $k \in \mathbb{N}$, such that $p_k \neq 0$ and*

$$\sum_{i=0}^{k} p_i(x) f^i(x) = 0 \ .$$

In other words, $P(x, f(x)) = 0$ for a nonzero polynomial $P = P(x, y) \in K[x, y]$.

In the previous proof we deduced D-finiteness of C from its algebraicity. We show that this transition works in general.

Proposition 3 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. If a* FPS $f \in K[[x]]$ *is algebraic then f is* D-*finite.*

Proof. Let $f \in K[[x]]$ be such that $\sum_{i=0}^{k} p_i f^i = 0$ where $k \in \mathbb{N}$, $p_i \in K[x]$ and $p_k \neq 0$. Thus

$$f^k = \sum_{i=0}^{k-1} q_i f^i \tag{1}$$

where $q_i \in K(x)$ are rational functions. Differentiating by x we get that

$$k f^{k-1} f' = \sum_{i=0}^{k-1} (q_i' f^i + q_i i f^{i-1} f') \text{ and } f' \left(k f^{k-1} - \sum_{i=0}^{k-1} q_i i f^{i-1} \right) = \sum_{i=0}^{k-1} q_i' f^i \ .$$

Thus

$$f' = \frac{P_1(f)}{Q_1(f)} \in K((x)) \tag{2}$$

(the field of formal Laurent series in x with coefficients in K) where $P_1(y)$ and $Q_1(y) \neq 0$ are polynomials in $K(x)[y]$. $Q_1(y) \neq 0$ because it has leading coefficient k. We show by induction on $j \in \mathbb{N}$ that there exist polynomials $P_j(y)$ and $Q_j(y) \neq 0$ in $K(x)[y]$ such that

$$f^{(j)} = \frac{P_j(f)}{Q_j(f)} \ .$$

For $j = 1$ we proved it above. We differentiate this equation by x and get that

$$f^{(j+1)} = \frac{(P_j(f))_x \cdot Q_j(f) - P_j(f) \cdot (Q_j(f))_x}{Q_j(f)^2} \ .$$

We have that $(P_j(f))_x = R_j(f) + S_j(f) f'$ and $(Q_j(f))_x = T_j(f) + U_j(f) f'$ where R_j, S_j, T_j and U_j are in $K(x)[y]$. Replacing f' by equation (2) we get the required expression

$$f^{(j+1)} = \frac{P_{j+1}(f)}{Q_{j+1}(f)} \text{ with } Q_{j+1} = Q_1 \cdot Q_j^2 \ .$$

We bring the $k+1$ fractions $f^{(0)} = \frac{f}{1}$, $f^{(1)} = \frac{P_1(f)}{Q_1(f)}$, $\ldots$, $f^{(k)} = \frac{P_k(f)}{Q_k(f)}$ to a common denominator, reduce powers of f in the numerators by equation (1) and get the expressions

$$f^{(0)} = \frac{A_0(f)}{B(f)}, \ f^{(1)} = \frac{A_1(f)}{B(f)}, \ \ldots, \ f^{(k)} = \frac{A_k(f)}{B(f)}$$

where $B(y) \neq 0$ and $A_i(y)$ are in $K(x)[y]$ and every $A_i(y)$ has in y degree at most $k-1$. It follows that the $k+1$ numerators can be non-trivially linearly combined to 0 by some coefficients in $K(x)$. Hence

$$\sum_{i=0}^{k} c_i f^{(i)} = 0$$

for some $c_i \in K(x)$, not all of them 0. Thus f is D-finite. □

In the initial proof we derived for $(C_n) = (C_1, C_2, \dots)$ the recurrence that for every $n \in \mathbb{N}$,

$$(n+1)C_{n+1} + (2-4n)C_n = 0 \ .$$

Recurrences of this form are characteristic for sequences of coefficients of D-finite formal power series.

Definition 4 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. We call a sequence $(a_n) = (a_0, a_1, \dots) \subset K$ P-recurrent (over K) if there exist polynomials $p_i \in K[x]$, where $i = 0, 1, \dots, k$ with $k \in \mathbb{N}_0$ and $p_k \neq 0$, such that for every $n \in \mathbb{N}_0$,*

$$\sum_{i=0}^{k} p_i(n) \cdot a_{n+i} = 0 \ .$$

It is easy to see that this is equivalent with the modified definition when the last displayed equality holds only for every $n > n_0$. To get (C_n) P-recurrent exactly according to Definition 4 we extend it with $C_0 := -\frac{1}{2}$.

Proposition 5 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $f = f(x) = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$. Then f is D-finite $\iff$ (a_n) is P-recurrent.*

Proof. Suppose that f is D-finite. For $i \in \mathbb{N}_0$ we introduce notation $(x)_0 = 0$ and $(x)_i = x(x-1)\dots(x-i+1)$ for $i > 0$. Thus f satisfies the differential equation

$$\sum_{i=0}^{k} p_i(x) f^{(i)}(x) = \sum_{(h,i)\in P} b_{h,i} x^h f^{(i)}(x) = 0 \qquad (3)$$

where $k \in \mathbb{N}_0$, $p_i \in K[x]$, $p_k \neq 0$, $P \subset \mathbb{N}_0^2$ is a nonempty set and every coefficient $b_{h,i} \in K$ is nonzero. Let $H \in \mathbb{N}_0$ be the maximum value of h in the pairs $(h, i) \in P$. Since $(i \in \mathbb{N}_0)$

$$f^{(i)}(x) = \sum_{n=0}^{\infty} (n+i)_i a_{n+i} x^n \ ,$$

by setting the coefficient of x^n, $n \in \mathbb{N}_0$, in equation (3) to zero we get for every $n \geq H$ that

$$\sum_{(h,i)\in P} b_{h,i} \cdot (n-h+i)_i \cdot a_{n-h+i} = 0 \ .$$

Grouping together summands sharing the same coefficient a_{n-h+i}, we deduce that the sequence (a_n) satisfies for $n > n_0$ a (nontrivial) P-recurrence as in Definition 4.

Suppose that the sequence (a_n) of coefficients of $f(x)$ is P-recurrent. Thus for every $n \in \mathbb{N}_0$ we have the equality

$$\sum_{i=0}^{k} p_i(n) \cdot a_{n+i} = 0$$

for some $p_i \in K[x]$ with $p_k \neq 0$. We switch from the basis $\{x^i \mid i \in \mathbb{N}_0\}$ of the K-vector space $K[x]$ to the basis $\{(x+i)_i \mid i \in \mathbb{N}_0\}$ and express every $p_i(x)$ as the linear combination

$$p_i(x) = \sum_{j=0}^{d_i} c_{i,j} \cdot (x+j)_j$$

where $d_i = \deg p_i \in \mathbb{N}_0$, $c_{i,j} \in K$ and $c_{k,d_k} \neq 0_K$ (if $p_i = 0$ then this sum is empty). Thus for every $n \in \mathbb{N}_0$,

$$\sum_{i=0}^{k} \left(\sum_{j=0}^{d_i} c_{i,j} \cdot (n+j)_j \right) a_{n+i} = 0 \ .$$

By grouping the terms according to the pairs (i,j) we get that for every $n \in \mathbb{N}_0$ the equality

$$\sum_{(i,j) \in Q} c_{i,j} \cdot (n+j)_j \cdot a_{n+i} = 0$$

holds, where $Q \subset \mathbb{N}_0^2$ is nonempty and (as we know) not all $c_{i,j} \in K$ are zero. It follows that

$$\sum_{(i,j) \in Q} c_{i,j} \cdot \left(x^{j-i} \cdot f(x) \right)^{(j)} = 0$$

because for every $n \in \mathbb{N}_0$ the coefficient of x^n in the FPS on the left side is 0. It is clear that this relation can be converted to a linear differential equation with polynomial coefficients, not all of them zero, for $f(x)$. We see that $f(x)$ is D-finite. □

D-finite formal power series were introduced by R. P. Stanley in [1]. For more information on uses of algebraic and D-finite generating functions in enumerative combinatorics see his book [2].

References

[1] R. P. Stanly, Differentiably finite power series, *European J. Combinatorics* **1** (1980), 175–188

[2] R. P. Stanly, *Enumerative Combinatorics. Volume 2*, Cambridge University Press, Cambridge, UK 1999