Lecture 7. The prime number theorem and the Riemann hypothesis. The Selberg–Delange method. Two arithmetic applications

M. Klazar

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In the seventh lecture we cover Chapter II.4. *The prime number theorem and the Riemann hypothesis*, Chapter II.5. *The Selberg–Delange method* and Chapter II.6. *Two arithmetic applications* in G. Tenenbaum's book [7], up to page 317.

Chapter II.4. The prime number theorem and the Riemann hypothesis

The following are Theorems 4.1 and 4.2 and Lemma 4.3 (Hadamard's three circles lemma) in [7].

Theorem 1 *For some $c > 0$ and every $x \geq 2$,*

$$\psi(x) = x + O\big(x \exp\big(-c\sqrt{\log x}\big)\big) \ \ \textit{and} \ \ \pi(x) = \mathrm{li}(x) + O\big(x \exp\big(-c\sqrt{\log x}\big)\big)\,.$$

This is the *Prime Number Theorem* with strong error term, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, $\pi(x) = \sum_{p \leq x} 1$ and $\mathrm{li}(x) = \int_2^x \mathrm{d}t/\log t$.

The *Riemann hypothesis* is the conjectured location of all nontrivial zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$. The *Lindelöf hypothesis* is the conjectured asymptotics $\zeta(\frac{1}{2} + i\tau) \ll_\varepsilon \tau^\varepsilon$ ($\tau \geq 2$). It is named after the Finnish mathematician *Ernst L. Lindelöf (1870–1946)*.

Theorem 2 *"The Riemann hypothesis implies that of Lindelöf. More precisely, if all non-trivial zeros of $\zeta(s)$ have real parts equal to $\frac{1}{2}$, then, for any $\varepsilon > 0$, we have*

$$\mathrm{Log}(\zeta(s)) \ll_\varepsilon (\log|\tau|)^{2-2\sigma+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \leq 1,\ |\tau| \geq 2)."$$

Here $\mathrm{Log}(r\exp(i\varphi)) = \log r + i\varphi$, $r > 0$ and $\varphi \in [-\pi, \pi)$.

Lemma 3 *If $F(s)$ is holomorphic in $0 < R_1 \leq |s| \leq R_2$ then the function*

$$M(r) = \max_{|s|=r} |F(s)|$$

is logarithmically convex on $[R_1, R_2]$.

This result is in fact due not to Hadamard but to Littlewood [5].

Finally, the following are Theorem 4.4, Corollary 4.5, Theorem 4.6. Lemmas 4.7 and 4.8 and Theorem 4.9 in [7].

Theorem 4 *Let* $\Theta := \inf(\{\xi > 0 \mid \psi(x) - x \ll x^{\xi}\ (x \geq 2)\})$. *Then*

$$\Theta = \sup(\{\beta > 0 \mid \zeta(\rho) = \zeta(\beta + i\gamma) = 0\})\,.$$

Corollary 5 *The Riemann hypothesis is equivalent to the asymptotics that for any* $\varepsilon > 0$, $\psi(x) = x + O_{\varepsilon}\bigl(x^{1/2+\varepsilon}\bigr)$ $(x \geq 2)$.

We define $\psi_0(x) := (\psi(x) + \psi(x-))/2$ and

$$\langle x\rangle := \min_{p,\nu\geq 1, p^{\nu}\neq x} \left|x - p^{\nu}\right|.$$

Theorem 6 *For* $x \geq 2$,

$$\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \log\left(1 - x^{-2}\right)/2$$

where we sum over non-trivial zeros $\rho = \beta + i\gamma$ *of* $\zeta(s)$ *so that* ρ *and* $\overline{\rho}$ *are paired. Also, for* $x, T \geq 2$ *one has that*

$$\psi_0(x) = x - \sum_{|\gamma|\leq T} \frac{x^{\rho}}{\rho} - \log(2\pi) - \log\left(1 - x^{-2}\right)/2 + R(x, T)$$

where $R(x,T) \ll (x/T)\log^2(xT) + (x\log x)/(x + T\langle x\rangle)$.

By [6], the second part of the theorem is due to H. von Koch [3] and E. Landau [4].

For $\kappa > 1$ and $T, x > 0$ we define

$$J_{\kappa}(x, T) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \frac{-\zeta'(s)x^{s}}{s\zeta(s)}\,\mathrm{d}s\,.$$

Lemma 7 *If* $x \geq 2$, $\kappa = 1 + 1/\log x$ *and* $T > 0$ *then*

$$|\psi_0(x) - J_{\kappa}(x, T)| \ll \frac{x\log^2 x}{T} + \frac{x\log x}{x + T\langle x\rangle}\,.$$

Lemma 8 *If* $\inf_{\rho=\beta+i\gamma} |\gamma - T| \gg 1/\log T$ *then for* $-1 \leq \sigma \leq 2$,

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log^2 T\,.$$

If $\min_{n\in\mathbb{N}_0} |s + 2m| \geq \frac{1}{2}$ *then for* $\sigma \leq -1$,

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log(2|s|)\,.$$

Theorem 9 *Let* $\Theta = \sup_{\rho=\beta+i\gamma} \beta$*. Then for* $x \ge 2$*,*

$$\psi(x) = x + O\big(x^{\Theta} \log^2 x\big) \ \text{ and } \ \pi(x) = \mathrm{li}(x) + O\big(x^{\Theta} \log x\big)\,.$$

Chapter II.5. The Selberg–Delange method

The following are Theorems 5.1, 5.2, 5.3 (Bateman, 1972) and 5.4 in [7]. The mentioned reference is [1].

Let $Z(s) = Z(s,z) = \big((s-1)\zeta(s)\big)^z/s\colon D \to \mathbb{C}$ where $D \subset \mathbb{C}$ is open, connected, avoids 0 and zeros of ζ, contains $[1,+\infty)$ and $Z(1,z) = 1$.

Theorem 10 $Z(s,z)$ *is holomorphic in the disc* $|s-1| < 1$ *where*

$$Z(s,\,z) = \sum_{j\ge 0} \frac{1}{j!}\gamma_j(z)(s-1)^j\,.$$

The coefficients $\gamma_j(z)$ *are entire function satisfying for any* $A, \varepsilon > 0$ *and* $|z| \le A$ *the bound*

$$\gamma_j(z)/j! \ll_{A,\varepsilon} (1+\varepsilon)^j\,.$$

Let $z \in \mathbb{C}$, $c_0 > 0$, $\delta \in (0,1]$ and $M > 0$. We say that $F(s) = \sum_{n\ge 1} a_n/n^s$ is $P(z, c_0, \delta, M)$ if $G(s,z) = F(s)\zeta(s)^{-z}$ has holomorphic extension to $\sigma \ge 1 - c_0/(1+\log^+|\tau|)$ and in this domain is bounded by

$$|G(s,z)| \le M(1+|\tau|)^{1-\delta}\,.$$

If $F(s)$ is $P(z,c_0,\delta,M)$ and there is $(b_n) \ge 0$ such that $|a_n| \le b_n$ and $\sum_{n\ge1} b_n/n^s$ is $P(w,c_0,\delta,M)$ then we say that $F(s)$ is $T(z,w,c_0,\delta,M)$.

Theorem 11 *Let* $F(s) = \sum_{n\ge1} a_n/n^s$ *be* $T(z,w,c_0,\delta,M)$*. For* $x \ge 3$*,* $N \ge 0$*,* $A > 0$*,* $|z|, |w| \le A$ *we have*

$$\sum_{n\le x} a_n = x(\log x)^{z-1}\bigg(\sum_{k=0}^{N} \frac{\lambda_k(z)}{(\log x)^k} + O\big(MR_N(x)\big)\bigg)$$

with $R_N(x) = R_N(x, c_1, c_2)$*. The constants* $c_1, c_2 > 0$ *and in* O *may depend only on* c_0*,* δ *and* A*.*

Theorem 12 *There is a* $c > 0$ *such that for every* $x \ge 1$*,*

$$|\{n \in \mathbb{N} \mid \varphi(n) \le x\}| = \frac{\zeta(2)\zeta(3)}{\zeta(6)}x + O\big(x \exp\big(-c\sqrt{\log x}\big)\big)\,.$$

We define $G(s,z) = F(s)\zeta(s)^{-z}$ ($F(s)$ is the given Dirichlet series) and

$$\lambda_k(z) = \frac{1}{\Gamma(z-k)} \sum_{h+j=k} \frac{1}{h!j!} G^{(h)}(1,\,z)\gamma_j(z)$$

where $\gamma_j(z)$ appear in Theorem 10.

Theorem 13 *Suppose that* $F(s) = \sum_{n \ge 1} a_n/n^s$ *converges for* $\sigma > 1$ *and that there is a* $z \in \mathbb{C}$ *and an* $N \in \mathbb{N}_0$ *such that* ($w = \max(\{1 - \mathrm{re}(z), 0\})$)

$$H_N(z) := \sum_{n=1}^{\infty} (|g_z(n)|/n)(\log(3n))^{N+1+w} < \infty\,.$$

Then for any z *with* $|z| \le A$,

$$\sum_{n \le x} = x(\log x)^{z-1} \Big(\sum_{k=0}^{N} \frac{\lambda_k(z)}{(\log x)^k} + O_A\big(H_N(z) R_N(x)\big) \Big)$$

with $R_N(x) = R_N(x, c_1, c_2)$ *where the constants* $c_1, c_2 > 0$ *may depend only on* A *and where the coefficients* $\lambda_k(z)$, $0 \le k \le N$, *are defined above.*

Chapter II.6. Two arithmetic applications

The following are Theorems 6.1–6.6 in [7]. Notation: $\omega(n) = \sum_{p \mid n} 1$, $A \in \mathbb{R}$, $x \in \mathbb{R}$, $z \in \mathbb{C}$,

Theorem 14 *For* $A > 0$ *there exist* $c_1 = c_1(A), c_2 = c_2(A) > 0$ *such that it holds uniformly for* $x \ge 3$, $|z| \le A$ *and* $N \in \mathbb{N}_0$ *that*

$$\sum_{n \le x} z^{\omega(n)} = x(\log x)^{z-1} \Big(\sum_{k=0}^{N} \frac{\lambda_k(z)}{(\log x)^k} + O_A(R_N(x)) \Big)$$

with

$$R_N(x) := \exp(-c_1 \sqrt{\log x}) + \big((c_2 N + 1)/\log x\big)^{N+1}$$

and

$$\lambda_k(z) := \frac{1}{\Gamma(z-k)} \sum_{h+j=k} \frac{1}{h!j!} G_1^{(h)}(1, z) \gamma_j(z)\,,$$

where the $\gamma_j(z)$ *are the entire functions defined in Theorem 10.*

Theorem 15 *For* $\delta \in (0,1)$ *there exist* $c_1 = c_1(\delta), c_2 = c_2(\delta) > 0$ *such that it holds uniformly for* $x \ge 3$, $|z| \le 2 - \delta$ *and* $N \in \mathbb{N}_0$ *that*

$$\sum_{n \le x} z^{\Omega(n)} = x(\log x)^{z-1} \Big(\sum_{k=0}^{N} \frac{\nu_k(z)}{(\log x)^k} + O_\delta(R_N(x)) \Big)$$

where $R_N(x)$ *is defined in the previous theorem and where*

$$\nu_k(z) := \frac{1}{\Gamma(z-k)} \sum_{h+j=k} \frac{1}{h!j!} G_2^{(h)}(1, z) \gamma_j(z)\,.$$

Theorem 16 *Let $A > 0$, $a_z(n)\colon \mathbb{N} \to \mathbb{C}$ with $z \in \mathbb{C}$ have in the disc $|z| \le A$ expansion $a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$. Let $h_j(z)$, $j = 0, 1, \ldots, N$, be holomorphic in $|z| \le A$ and the quantity $R_N(x)$ (independent of z) be such that for $x \ge 3$ and $|z| \le A$ we have*

$$\sum_{n \le x} a_z(n) = x(\log x)^{z-1}\biggl(\sum_{j=0}^{N} \frac{z h_j(z)}{(\log x)^j} + O_A(R_N(x))\biggr).$$

Then it holds uniformly for $x \ge 3$ and $1 \le k \le A \log\log x$ that $C_k(x) := \sum_{n \le x} c_k(n)$ is

$$\frac{x}{\log x}\biggl(\sum_{j=0}^{N} \frac{Q_{j,k}(\log\log x)}{(\log x)^j} + O_A\biggl(\frac{(\log\log x)^k}{k!} R_N(x)\biggr)\biggr),$$

where ($0 \le j \le n$ and $k \in \mathbb{N}$)

$$Q_{j,k}(X) := \sum_{m+k=k-1} \frac{1}{m!l!} h_j^{(m)}(0) X^l .$$

In addition, if $|h_0''(z)| \le B$ for $|z| \le A$ then it holds uniformly for $x \ge 3$ and $1 \le k \le A \log\log x$ that $C_k(x)$ is

$$\frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\biggl(h_0\biggl(\frac{k-1}{\log\log x}\biggr) + O_A\biggl(\frac{(k-1)B}{(\log\log x)^2} + \frac{\log\log x}{k} R_0(x)\biggr)\biggr).$$

Let $\pi_k(x) = |\{n \le x \mid \omega(n) = k\}|$ and $N_k(x) = |\{n \le x \mid \Omega(n) = k\}|$. Let

$$\lambda(z) = \frac{1}{\Gamma(z+1} \prod_p (1 + z/(p-1))(1-1/p)^z$$

and

$$\nu(z) = \frac{1}{\Gamma(z+1} \prod_p (1 - z/)^{-1}(1-1/p)^z .$$

Theorem 17 *For $A > 0$ there exist $c_1 = c_1(A), c_2 = c_2(A) > 0$ such that it holds uniformly for $x \ge 3$, $1 \le k \le A \log\log x$ and $N \in \mathbb{N}_0$ that*

$$\pi_k(x) = \frac{x}{\log x}\biggl(\sum_{j=0}^{N} \frac{P_{j,k}(\log\log x)}{(\log x)^j} + O\biggl(\frac{(\log\log x)^k}{k!} R_N(x)\biggr)\biggr)$$

where $P_{j,k} \in \mathbb{R}[X]$ has degree $\le k-1$ and $R_N(x)$ is defined in Theorem 14. In particular, we have

$$P_{0,k}(x) = \sum_{m+l=k-1} \frac{1}{m!l!} \lambda^{(m)}(0) X^l .$$

Moreover we have under the same assumptions that

$$\pi_k(x) = \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\bigl(\lambda\bigl((k-1)/\log\log x\bigr) + O\bigl(k/(\log\log x)^2\bigr)\bigr).$$

Theorem 18 *For $\delta \in (0,1)$ there exist $c_1 = c_1(\delta), c_2 = c_2(\delta) > 0$ such that it holds uniformly for $x \geq 3$, $1 \leq k \leq (2-\delta)\log\log x$ and $N \in \mathbb{N}_0$ that*

$$N_k(x) = \frac{x}{\log x}\biggl(\sum_{j=0}^{N} \frac{Q_{j,k}(\log\log x)}{(\log x)^j} + O\biggl(\frac{(\log\log x)^k}{k!} R_N(x)\biggr)\biggr)$$

where $Q_{j,k} \in \mathbb{R}[X]$ has degree $\leq k-1$ and $R_N(x)$ is defined in Theorem 14. In particular, we have

$$Q_{0,k}(x) = \sum_{m+l=k-1} \frac{1}{m!l!}\nu^{(m)}(0)X^l\,.$$

Moreover we have under the same assumptions that

$$N_k(x) = \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\bigl(\nu\bigl((k-1)/\log\log x\bigr) + O\bigl(k/(\log\log x)^2\bigr)\bigr)\,.$$

Theorem 19 *Let $\delta \in (0,1)$, $A > 0$ and $C = \frac{1}{4}\prod_{p>2}(1+\frac{1}{p(p-2)}) \approx 0.378694$. Then it holds uniformly for $x \geq 3$ and $(2+\delta)\log\log x \leq k \leq A\log\log x$ that*

$$N_k(x) = \frac{Cx\log x}{2^k}\bigl(1 + O\bigl((\log x)^{-\delta^2/5}\bigr)\bigr)\,.$$

Finally, there are Theorems 6.7 and 6.8 in [7]; $\tau(n)$ is the number of divisors of n. Further, for $u \in [0,1]$ we define

$$F_n(u) = \frac{1}{\tau(n)} \sum_{\substack{d\,|\,n\\ d \leq n^u}} 1\,.$$

Theorem 20 *It holds uniformly for $x \geq 2$ and $u \in [0,1]$ that*

$$\frac{1}{x}\sum_{n\leq x} F_n(u) = \frac{2}{\pi}\arcsin\sqrt{u} + O\bigl(1/\sqrt{\log x}\bigr)\,.$$

This theorem follows from the next one.

Theorem 21 *Let $h = \prod_p \sqrt{p(p-1)}\log(1/(1-p)) \approx 0.969$. It holds uniformly for $x \geq 2$ and $d \in \mathbb{N}$ that*

$$\sum_{n\leq x} \frac{1}{\tau(nd)} = \frac{hx}{\sqrt{\pi\log x}}\bigl(g(d) + O\bigl((3/4)^{\omega(d)}/\log x\bigr)\bigr)$$

where $g\colon \mathbb{N} \to \mathbb{R}$ satisfies that $\sum_{d\leq x} g(d) = \frac{x}{h\sqrt{\pi\log x}}(1 + O(1/\log x))$.

The last two theorems were obtained in [2].

References

[1] P. Bateman, The distribution of values of the Euler function, *Acta Arith.* **21** (1972), 329–345

[2] J. M. Deshouillers, F. Dress and G. Tenenbaum, Lois de répartition des diviseurs, *Acta Arith.* **23** (1979), 273–285

[3] H. von Koch, Contributions à la théorie des nombres premiers, *Acta Math.* **33** (1910), 293–320

[4] E. Landau, Über einige Summen, die von den Nullstellen der Riemanschen Zetafunktion abhängen, *Acta Math.* **35** (1912), 271–294

[5] J. E. Littlewood, Quelques consequences de l'hypothese que la function $\zeta(s)$ de Riemann n'a pas de zeros dans le demi-plan $\mathrm{Re}(s) > 1/2$, *Les Comptes rendus de l'Académie des sciences*, **154** (1912) 263–266

[6] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press, Cambridge, UK 2007

[7] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)

[8] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford 1986 (Second Edition, revised by D. R. Heath-Brown)