Lecture 1. Summation formulae. Prime numbers. Arithmetic functions

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In my lectures I will survey in detail the textbook [9] *Introduction to Analytic and Probabilistic Number Theory* written by Gérald Tenenbaum. I will mention every Definition, Lemma, Proposition, Corollary and Theorem in the book. For time reasons I cannot cover the sections of (historical) Notes and Exercises. I will prove only tiny selection of results in the book but I do want to prove at least one result in each of the 22 chapters.

I am faithful to the notation used in the book, but not dogmatically. Thus I replace ln and $\ln_2$ with log and log log, and $[0,+\infty[$ (and the like) with $[0,+\infty)$. I often shorten and abridge statements of theorems.

Let $\mathbb{N}=\{1,2,\dots\}$ and $\mathbb{N}_0=\{0,1,\dots\}$, $\mathbb{Z}$ be the integers and $\mathbb{R}$ and $\mathbb{C}$ be the real and the complex numbers. The letters k, l, m and n range in $\mathbb{N}$ and $x,y\in\mathbb{R}$ and p denotes a prime number. By $k\,|\,l$ we denote the divisibility relation on $\mathbb{Z}$. Divisors of $n\in\mathbb{N}$ are always positive. For $x\in\mathbb{R}$, $\lfloor x\rfloor=\max(\mathbb{Z}\cap(-\infty,x])$ is the lower integer part of x; the upper integer part $\lceil x\rceil$ is defined similarly. For any finite set X we denote by $|X|\in\mathbb{N}_0$ the number of elements in X.

In the first lecture we cover Chapter I.0. *Some tools from real analysis*, Chapter I.1. *Prime numbers* and Chapter I.2. *Arithmetic functions*, up to page 43.

Chapter I.0. Some tools from real analysis

The following is Theorem 0.1 (Abel's transformation) in [9].

Theorem 1 *If $(a_n),(b_n)\subset\mathbb{C}$ $(n=0,1,\dots)$ then for any $N\in\mathbb{N}_0$ and $M\in\mathbb{N}$,*

$$\sum_{N<n\le N+M}a_nb_n=A_{N+M}b_{N+M+1}+\sum_{N<n\le N+M}A_n(b_n-b_{n+1})\ ,$$

where $A_n:=\sum_{N<m\le n}a_m$ $(n\ge 0)$. In particular, if

$$\sup_{N<n\le N+M}|A_n|\le A\ ,$$

and if (b_n) *is non-negative and non-increasing, then*

$$\left|\sum_{N<n\le N+M} a_n b_n\right| \le A b_{N+1}\ .$$

The following is Corollary 0.2 (Abel's convergence criterion or Abel's rule) in [9].

Corollary 2 *Let* $(a_n)\subset\mathbb{C}$*,* $(b_n)\subset\mathbb{R}^+$ *be non-increasing* $(n=0,1,\dots)$ *and let*

$$\lim_{n\to\infty} b_n = 0 \ \ \textit{and}\ \ \sup_{N\ge 0}\left|\sum_{0\le n\le N} a_n\right| \le A\ .$$

Then the series $\sum_{n\ge 0} a_n b_n$ *converges, and for every* $N\in\mathbb{N}_0$ *we have*

$$\left|\sum_{n>N} a_n b_n\right| \le 2A b_{N+1}\ .$$

From now the *Stieltjes integral* is being employed and [9] refers for it to the book [11]. At the end of this Chapter I.0 I briefly review the definition. The following *Abel's summation formula* is Theorem 0.3 in [9]. Recall the $\mathcal{C}^k$ notation for sets of k times continuously differentiable functions.

Theorem 3 *Let* $(a_n)\subset\mathbb{C}$ $(n=1,2,\dots)$ *and let*

$$A(t) := \sum_{n\le t} a_n \qquad (t>0).$$

Then, for any function $b\in\mathcal{C}^1([1,x])$*, we have*

$$\sum_{1\le n\le x} a_n b(n) = A(x)b(x) - \int_1^x A(t)\, b'(t)\,\mathrm{d}t.$$

Proof. In [9] this is proven via integration by parts in Stieltjes integrals (the measure $\mathrm{d}A(t)$ appears). I take the integrals to be Riemann and prove the identity by the additivity device which I learned in [10].

So we prove the more general identity

$$\sum_{m<n\le x} a_n b(n) = A(x)b(x) - A(m)b(m) - \int_m^x A(t)\, b'(t)$$

where $m\in\mathbb{N}$, $m<x$, $A(t)$ is as above and $b\in\mathcal{C}^1([m,x])$ (in fact, mere differentiability of b on $[m,x]$ suffices). We partition the interval $(m,x]$ in the subintervals $(m,m+1]\cup(m+1,m+2]\cup\dots\cup(\lfloor x\rfloor,x]$ and observe that each side of the identity is additive in this partition (the value of the side over $(m,x]$ equals to the sum of its values over the subintervals). Thus it suffices to prove the

identity only for $x \le m+1$. The right side then becomes, by the Fundamental Theorem of Calculus,

$$A(x)b(x) - A(m)b(m) - A(m)\int_m^x b'(t) = (A(x) - A(m))b(x).$$

If $x < m+1$ then the last expression is $(A(m)-A(m))b(x) = 0$, and if $x = m+1$ then it is $(A(m+1) - A(m))b(x) = a_{m+1}b(m+1)$. In both cases it agrees with the value of the sum on the left side of the identity. □

The following is Theorem 0.4 (Comparison of a sum and an integral) in [9].

Theorem 4 *Let* $a < b$ *be in* $\mathbb{Z}$ *and* $f\colon [a,b] \to \mathbb{R}$ *be monotonic. Then for some* $\vartheta \in [0,1]$,

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\,\mathrm{d}t + \vartheta\big(f(b) - f(a)\big).$$

Proof. In [9] this is again proven via integration by parts in Stieltjes integrals (the measure $\mathrm{d}\lfloor t\rfloor$ appears). I give a simpler proof by means of Riemann integrals.

We denote the displayed sum by S. Suppose that f weakly decreases, the other case is similar. For $n \in \mathbb{N} \cap (a,b]$ we have that $f(n-1) \ge \int_{n-1}^n f \ge f(n)$. We sum these bounds over the mentioned n and get (by the additivity of integrals) the bound

$$S + f(a) - f(b) \ge \int_a^b f \ge S.$$

Thus indeed $\int_a^b f \ge S \ge \int_a^b f + f(b) - f(a)$, as required. □

The following is Corollary 0.5 in [9].

Corollary 5 *For* $n \ge 1$, *we have* $\log n! = n\log n - n + 1 + \vartheta\log n$, *with* $\vartheta = \vartheta_n \in [0,1]$.

The following is Theorem 0.6 (Second mean value theorem) in [9].

Theorem 6 *Let* $a < b$ *be in* $\mathbb{R}$, $f\colon [a,b] \to \mathbb{R}$ *be monotonic and* $g\colon [a,b] \to \mathbb{R}$ *be integrable. Then for some* $\xi \in [a,b]$,

$$\int_a^b f(t)g(t)\,\mathrm{d}t = f(a)\int_a^\xi g(t)\,\mathrm{d}t + f(b)\int_\xi^b g(t)\,\mathrm{d}t.$$

The following is Theorem 0.7 (Euler–Maclaurin summation formula) in [9]. The *Bernoulli polynomials* $b_r(x) \in \mathbb{Q}[x]$ and the *Bernoulli numbers* $B_r = b_r(0)$, $r \in \mathbb{N}_0$, are defined by the expansions

$$\sum_{r=0}^{\infty} b_r(x)\cdot\frac{y^r}{r!} = \frac{y\cdot \mathrm{e}^{xy}}{\mathrm{e}^y - 1} \text{ and } \sum_{r=0}^{\infty}\frac{B_r y^r}{r!} = \frac{y}{\mathrm{e}^y - 1}.$$

So $B_{2i+1} = 0$ for $i \in \mathbb{N}$ and (as stated in [9])

$$\begin{aligned}&(B_0,\, B_1,\, B_2,\, B_4,\, B_6,\, B_8,\, B_{10},\, B_{12},\, B_{14},\, B_{16},\, \dots)\\ &= \left(1,\, -\tfrac{1}{2},\, \tfrac{1}{6},\, -\tfrac{1}{30},\, \tfrac{1}{42},\, -\tfrac{1}{30},\, \tfrac{5}{66},\, -\tfrac{691}{2730},\, \tfrac{7}{6},\, -\tfrac{3617}{510},\, \dots\right).\end{aligned}$$

The function $B_r(x)\colon \mathbb{R} \to \mathbb{R}$ is defined as the 1-periodic extension of the restriction $b_r(x) \mid [0, 1)$.

Theorem 7 *If $k \in \mathbb{N}_0$, $a < b$ are in $\mathbb{Z}$ and $f \in \mathcal{C}^{k+1}([a, b])$, then*

$$\begin{aligned}\sum_{a<n\le b} f(n) = \int_a^b f(t)\,\mathrm{d}t \quad &+ \sum_{0\le r\le k} \frac{(-1)^{r+1} B_{r+1}}{(r+1)!}(f^{(r)}(b) - f^{(r)}(a))\\ &+ \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t) f^{(k+1)}(t)\,\mathrm{d}t.\end{aligned}$$

The following is Theorem 0.8 in [9].

Theorem 8 *For $n \ge 1$, we have*

$$\sum_{m\le n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\vartheta}{60n^4},$$

where γ is Euler's constant and $\vartheta = \vartheta_n \in [0, 1]$.

The Stieltjes integral

We define it after [7, Appendix A]; in [7] it is called the Riemann–Stieltjes integral. Let $a < b$ be in $\mathbb{R}$ and $f, g\colon [a, b] \to \mathbb{R}$. For any tagged partition $P = (\overline{a}, \overline{b})$ of $[a, b]$, in which $\overline{a} = (a = a_0 < a_1 < \dots < a_n = b)$, $n \in \mathbb{N}$, and the tags are $b_i \in [a_{i-1}, a_i]$, we define (the Stieltjes sum)

$$S(f,\, g,\, P) = \sum_{i=1}^{n} f(b_i) \cdot (g(a_i) - g(a_{i-1})).$$

We also set $\Delta(P) = \max(\{a_i - a_{i-1} \mid i = 1, \dots, n\})$.

Definition 9 (the Stieltjes $\int$) *Let a, b, f and g be as above. If there exists an $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every tagged partition P of $[a, b]$ with $\Delta(P) < \delta$ it holds that*

$$\left|S(f,\, g,\, P) - I\right| < \varepsilon,$$

we say that I is the Stieltjes integral of f over $[a, b]$ with respect to g and denote it by

$$\int_a^b f\,\mathrm{d}g \quad (:= I).$$

Here is the basic existence theorem for SI as given (and proved) in [7].

Theorem 10 $\int_a^b f \mathrm{d}g$ *exists if* $f \in \mathcal{C}([a,b])$ *and* g *has bounded variation on* $[a,b]$.

The last condition on g means that there exists a $c > 0$ such that for any tuple $a_0 < a_1 < \cdots < a_n$ in $[a,b]$,

$$\sum_{i=1}^{n} |g(a_i) - g(a_{i-1})| < c.$$

A few more theorems on SI are stated and proven in [7] but we do not mention them here. The attractive feature of SI is that it encompasses discrete sums of the form we encountered above: if f is continuous on $[a,b]$ then

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\,\mathrm{d}\lfloor t \rfloor.$$

Chapter I.1. Prime numbers

The following is Theorem 1.1 (Fundamental theorem of arithmetic) in [9].

Theorem 11 *Each natural number* > 1 *can be represented in a unique way, up to the order of the factors, as a product of prime numbers.*

The following is Theorem 1.2 in [9]; $\pi(x)$ is the number of primes not exceeding x.

Theorem 12 *We have*

$$\pi(x) > \frac{\log\log x}{\log 2} - \frac{1}{2} \qquad (x \ge 2).$$

The following Theorem 1.3 in [9] is an explicit and thus stronger form of the result $\frac{x}{\log x} \ll \pi(x) \ll \frac{x}{\log x}$ obtained by P. L. Chebyshev in [3] in 1850 (I found this reference in [7]).

Theorem 13 *For* $n \ge 4$*, we have*

$$(\log 2)\frac{n}{\log n} \le \pi(n) \le \left(\log 4 + \frac{8\log\log n}{\log n}\right)\frac{n}{\log n}.$$

The proof (given in [9]) of the following Theorem 1.4 in [9] “was found independently by Erdős and Kalmár in 1939.”.

Theorem 14 *For* $n \ge 1$*, we have*

$$\prod_{p\le n} p \le 4^n.$$

The following is Theorem 1.5 (Nair) in [9]; d_n is the least common multiple of the numbers $1, 2, \ldots, n$ and the reference is to the article [8].

Theorem 15 *For $n \geq 7$, we have $d_n \geq 2^n$.*

The following is Theorem 1.6 in [9]; $\upsilon_p(n) := k \in \mathbb{N}_0$ such that $p^k \mid n$ but p^{k+1} does not divide n. This result is due to A.-M. Legendre [4].

Theorem 16 *For each prime number p, we have*

$$\upsilon_p(n!) = \sum_{k \geq 1} \lfloor n/p^k \rfloor \qquad (n \geq 1).$$

The following is Corollary 1.7 in [9].

Corollary 17 *For each prime p, we have*

$$\frac{n}{p} - 1 < \upsilon_p(n!) \leq \frac{n}{p} + \frac{n}{p(p-1)} \qquad (n \geq 1).$$

The following Theorem 1.8 (Mertens' first theorem) in [9] is (by [7]) due to F. Mertens [5, 6].

Theorem 18 *For $x \geq 2$, we have*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Moreover, the $O(1) \in (-1 - \log 4, \log 4)$.

Proof. Let us prove simplified Mertens' first theorem, without the restriction on the $O(1)$. For $x \geq 2$ and $n = \lfloor x \rfloor$,

$$\begin{aligned} n \log n + O(n) &\overset{\text{Cor. 5}}{=} \log(n!) \\ &= \sum_{p \leq x} \upsilon_p(n!) \log p \\ &\overset{\text{prev. cor.}}{=} n \sum_{p \leq x} \frac{\log p}{p} + O(n) \sum_{p \leq x} \frac{\log p}{p(p-1)} \\ &= n \sum_{p \leq x} \frac{\log p}{p} + O(n). \end{aligned}$$

Dividing by n and using that $\log n = \log x + O(1/x)$ we get the result. □

The following is Theorem 1.9 in [9].

Theorem 19 *Set* $c_0 := \sum_p \big(\log(1/(1-1/p)) - \frac{1}{p}\big) \approx 0.315718$*. Then we have, for* $x \geq 2$*,*

$$\sum_{p \leq x} \frac{1}{p} = \log\left(1/\prod_{p \leq x}\left(1-\frac{1}{p}\right)\right) - c_0 + \frac{\vartheta}{2(x-1)}$$

where $\vartheta = \vartheta(x) \in (0,1)$*.*

The following Theorem 1.10 in [9] is (by [7]) due to F. Mertens [5, 6].

Theorem 20 *There is a constant* c_1 *such that, for* $x \geq 2$*,*

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + c_1 + O(1/\log x).$$

In addition, the constant involved in the Landau symbol can be chosen $\leq 2(1 + \log 4) < 5$*.*

The following is Theorem 1.11 in [9]; $\mathrm{e} = 2.71828\ldots$ is the Euler number.

Theorem 21 *With the constants* c_0 *and* c_1 *as in Theorems* 1.9 *and* 1.10*, we have, for* $x \geq 2$*,*

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right) = \frac{\mathrm{e}^{-(c_0+c_1)}}{\log x}\big(1 + O(1/\log x)\big).$$

The following Theorem 1.12 (Mertens formula) in [9] is (by [7]) due to F. Mertens [5, 6]. The constants c_0 and c_1 are as in the two previous theorems.

Theorem 22 *We have* $c_0 + c_1 = \gamma$*, where* γ *denotes Euler's constant. Thus*

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right) = \frac{\mathrm{e}^{-\gamma}}{\log x}\big(1 + O(1/\log x)\big) \qquad (x \geq 2).$$

The following Theorem 1.13 in [9] was (by [7]) obtained by P. L. Chebyshev in [2] in 1848 by using the zeta function $\zeta(s)$; the proof in [9] uses Theorem 1.10 (here Theorem 20).

Theorem 23 *We have,*

$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \leq 1 \leq \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}.$$

Thus if $\lim_{x\to+\infty} \pi(x)/(x/\log x)$ exists then it must be 1.

Chapter I.2. Arithmetic functions

The following is Theorem 2.1 in [9]; $\tau(n)$ is the number of divisors of n and $\|$ denotes the maximum divisibility by a prime power. *Multiplicativity* of $f\colon \mathbb{N} \to \mathbb{C}$ means that $f(1) = 1$ and $f(mn) = f(m)f(n)$ if m and n are coprime.

Theorem 24 *The divisor function is multiplicative. We have*

$$\tau(n) = \prod_{p^\nu \,\|\, n} (\nu + 1) \qquad (n \geq 1).$$

The following is Theorem 2.2 in [9]; the *Möbius function* $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ has values $\mu(n) = 0$ if n is not square-free and $\mu(n) = (-1)^k$ if n is a product of k distinct primes.

Theorem 25 *The Möbius function is multiplicative.*

The following is Definition 2.3 in [9]; arithmetic functions are functions of the type $f\colon \mathbb{N} \to \mathbb{C}$.

Definition 26 *Let f be an arithmetic function. The formal Dirichlet series associated to f is the formal series*

$$D(f; s) := \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

The following is Theorem 2.4 in [9]. $\mathbb{A} = (A, 0_A, 1_A, +, *)$ is the (commutative unital) ring on the set A of arithmetic functions, in which 0_A is the zero function, 1_A is 1 on 1 and 0 elsewhere, $+$ is the pointwise addition and $*$ is the *Dirichlet convolution*

$$(f * g)(n) = \textstyle\sum_{d \,|\, n} f(d)g(n/d)\ .$$

Recall that units in a ring are the invertible elements. E. D. Cashwell and C. J. Everett proved in [1] in 1959 that the ring $\mathbb{A}$ is factorial, enjoys unique factorization in irreducibles.

Theorem 27 *The group $\mathbb{G}$ of units in the ring $\mathbb{A}$ of arithmetic functions consists of those arithmetic functions f with $f(1) \neq 0$.*

The following is Theorem 2.5 in [9].

Theorem 28 *$f \in A$ is multiplicative iff*

$$D(f; s) = \prod_p \left(1 + \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^{\nu s}}\right).$$

Let $M \subset A$ be the set of multiplicative arithmetic functions. The following is Theorem 2.6 in [9].

Theorem 29 *M is a subgroup of the group of units in $\mathbb{A}$.*

The following is Theorem 2.7 in [9]; $\sigma(n) = \sum_{d \mid n} d$.

Theorem 30 *The function $\sigma(n)$ is multiplicative.*

The following is Theorem 2.8 in [9]; recall that μ is the Möbius function and 1_A is the identity in $\mathbb{A}$. By $\mathbf{1} \in A$ we denote the constantly 1 function.

Theorem 31 *As $\mathbf{1} * \mu = 1_A$, the Möbius function is the (convolutional) inverse of $\mathbf{1}$. Explicitly, $\sum_{d \mid n} \mu(d)$ is 1 if $n = 1$ and 0 if $n > 1$.*

The following is Theorem 2.9 (First Möbius inversion formula) in [9]; the variable n ranges in $\mathbb{N}$.

Theorem 32 *If $f, g \in A$ then*

$$\forall\, n \left(g(n) = \textstyle\sum_{d \mid n} f(n)\right) \iff \forall\, n \left(f(n) = \textstyle\sum_{d \mid n} g(d)\mu(n/d)\right).$$

The following is Theorem 2.10 (Second Möbius inversion formula) in [9]; the variable x ranges in $[1, +\infty)$.

Theorem 33 *If $F, G\colon [1, +\infty) \to \mathbb{R}$ then*

$$\forall\, x \left(G(x) = \textstyle\sum_{n \le x} F(x/n)\right) \iff \forall\, x \left(F(x) = \textstyle\sum_{n \le x} \mu(n)G(x/n)\right).$$

The following is Theorem 2.11 in [9]. The *von Mangoldt function* $\Lambda = \mu * \log\colon \mathbb{N} \to [0, +\infty)$ has values $\Lambda(n) = 0$ if $n \in \mathbb{N}$ is not a prime power and $\Lambda(p^\nu) = \log p$. We have the summatory functions

$$\psi(x) := \sum_{n \le x} \Lambda(n) \text{ and } \vartheta(x) := \sum_{p \le x} \log p$$

which were introduced by P. L. Chebyshev. Recall that $\pi(x)$ counts prime numbers $\le x$.

Theorem 34 *For $x \ge 2$, we have*

$$\psi(x) = \vartheta(x) + O(\sqrt{x}) \text{ and } \pi(x) = \frac{\vartheta(x)}{\log x} + O\big(x/(\log x)^2\big).$$

The following is Corollary 2.12 in [9].

Corollary 35 *Let $\alpha \in (0, \log 2)$ and $\beta > \log 4$. For large enough x, we have*

$$\alpha x \le \vartheta(x) \le \psi(x) \le \beta x.$$

Finally, the following is Theorem 2.13 in [9]. One defines *Euler's totient function* $\varphi\colon \mathbb{N} \to \mathbb{N}$ by $\varphi(n) := |\{m \mid m \le n, (m,n) = 1\}|$; $\varphi(n)$ counts the natural numbers m not exceeding n and coprime to n.

Theorem 36 *The function φ is multiplicative and for every $n \in \mathbb{N}$,*

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

Proof. Three proofs are indicated in [9] and we remind the last one which utilizes the *principle of inclusion and exclusion* (PIE). For $m, n \in \mathbb{N}$ we denote $[n] = \{1, 2, \dots, n\}$ and $A_m = \{k \in [n] \mid m \mid k\}$. We set $P_n = \{p \mid p \mid n\}$. By PIE,

$$\varphi(n) = \left|[n] \setminus \bigcup_{p \in P_n} A_p\right| = \sum_{X \subset P_n} (-1)^{|X|} \left|\bigcap_{p \in X} A_p\right|$$

where for $X = \emptyset$ we interpret the intersection as $[n]$. Since $|A_q| = n/q$ whenever q is a product of some primes in P_n and for $p \neq p'$ it holds that $p \mid n \wedge p' \mid n \iff pp' \mid n$, the last sum equals

$$\sum_{X \subset P_n} (-1)^{|X|} \left(n \prod_{p \in X} \frac{1}{p}\right) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$

where for $X = \emptyset$ and $n = 1$ the products are defined to be 1. □

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