Homework assignment, 18. 3.

Problem 1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with real coefficients. We say that a power series $B(x)$ is a *square root* of $A(x)$ if $B(x)^2 = A(x)$. What conditions must the coefficients of $A(x)$ satisfy in order for $A(x)$ to have a square root? And if $A(x)$ satisfies these conditions, how many square roots does it have?

Problem 2. Suppose that we toss a fair coin, and we record whether the outcome was heads ('H') or tails ('T'), where both outcomes have the same probability $\frac{1}{2}$. Tossing the coin repeatedly independently and recording the sequence of outcomes, we obtain a random potentially infinite sequence $C_1, C_2, C_3, \dots$, where each C_i is a random element of the set $\{H, T\}$.

Suppose now that two players, Alice and Bob, are playing the following game: with a sequence of coin tosses $C_1, C_2, \dots$ as above, Alice waits for the first occurrence of the pattern HTH, i.e., a sequence of three consecutive coin tosses in which the first and the third are H, and the middle one is T. Meanwhile, Bob waits for the first occurrence of the pattern THH. The sequence of coin tosses continues until either HTH or THH appears, and the player whose pattern has appeared is then declared the winner. What is the probability that Alice wins? Would the probability be different if Bob waited for HHT instead of THH?

Problem 3. An *integer composition of size n with k parts* is an ordered sequence of positive integers $m_1, m_2, \dots, m_k$ such that $n = m_1 + m_2 + \cdots + m_k$. Observe that for $n \geq 1$, there are 2^{n-1} integer compositions of size n (with any number of parts). Suppose we choose one such composition of size n uniformly at random. What is the expected number of parts in such a random composition? And what is the expected number of parts that are greater than 1?

Problem 4. Let $g_n = 2^{\binom{n}{2}}$ be the number of graphs on the vertex set $\{1, 2, \dots, n\}$, and let c_n be the number of such graphs that are connected. Let us put $g_0 = 1$ and $c_0 = 0$ by convention. What is the relationship between the corresponding EGFs $G(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$ and $C(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$? Can you find an algorithm to determine the value c_n exactly, with time complexity polynomial in n?