Third recitation session, April 4

We looked at various problems related to the exponential generating functions of permutations.

Problem 1. Let c_n be the number of permutations of the set $[n] = \{1, \dots, n\}$ that have exactly one cycle, and let $C(x) = \sum_{n\geq 0} a_n x^n/n!$ be the corresponding EGF. Find an explicit formula for c_n. (By the way, are you able to calculate the value of the sum $C(x)$ as a function of x, and find the analytic function having the series $C(x)$? This last question is outside the scope of the course and may require some knowledge of calculus. The answer is that $C(x) = -\ln(1-x)$.)

Problem 2. For an integer k, let $D_k(x)$ be the EGF of permutations having exactly k cycles. In particular, we have $D_1(x) = C(x)$. We adopt the convention that $D_0(x) = 1$, corresponding to the fact that the empty permutation is the only permutation with no cycles. Express $D_k(x)$ as a function of $C(x)$.

Problem 3. Define the series $P(x) = \sum_{k\geq 0} D_k(x)$ and observe that $P(x)$ is the EGF of the class of all permutations. Using the results of previous problems, simplify the expression $\sum_{k\geq 0} D_k(x)$ and check that $P(x) = 1/(1-x)$, which is the expected result, corresponding to the fact that there are exactly $n!$ permutations of $[n]$.

Problem 4. Let $O(x)$ be the EGF of the class of permutations with an odd number of cycles and let $E(x)$ be the analogous EGF for even number of cycles. Express $O(x)$ and $E(x)$ as a function of $C(x)$. Simplify the expressions for $O(x)+E(x)$ and check that you get $P(x)$. Simplify the expression for $E(x)-O(x)$ and check that you get $1-x$, and explain by a combinatorial argument why this is the expected result.

Problem 5. Fix an integer d. Let a_n be the number of permutations that have no cycle of length d, and let $P^*(x) = \sum_{d\geq 0} a_n x^n/n!$ be the corresponding EGF. Find a formula for $P^*(x)$, by adapting the results of the previous problems suitably.

Problem 6. Fix again an integer d. Let $b_{n,k}$ be the number of permutations of $[n]$ that have exactly k cycles of length d. Consider the power series $Q(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} b_{n,k} y^k x^n/n!$. Note that this is the weighted EGF for the class of permutations with each permutation π having weight $w(\pi) = y^m$ where m is the number of d-cycles in π. Find a formula for $Q(x,y)$. Substitute $y=1$ and $y=0$ into the formula for $Q(x,y)$ and check that you obtain the expected results (what are the expected results?).