Analytic Combinatorics — Final Exam

Submit your solutions no later than end of February 2013, either by email (jelinek@iuuk.mff.cuni.cz) or in person. Your solutions may apply any method or technique you consider appropriate, not just those that were presented at the lecture. Full credit will be given to any solution that yields a result which is as good as the result obtainable by the methods from the lecture. If you derive an even stronger result or if your method is particularly creative, you may receive bonus points on top of the full credit. You may submit partial solutions or corrections of previous solutions as often as you wish, as long as you observe the deadline specified above.

Grades will be awarded as follows:

points	0–7	8–10	11–13	14–∞
grade	4 (fail)	3 (good)	2 (very good)	1 (excellent)

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

You may use without proof the Stirling approximation of $n!$ and similar standard results. You may also use without proof any result from the lecture.

Question 1. Let $\mathcal{W}_n$ be the set of words of length n over the alphabet {A,B} in which no two consecutive letters are equal to B. Let $\mathcal{W} = \bigcup_{n\geq 0} \mathcal{W}_n$. For a word $w \in \mathcal{W}$, the *initial A-run* of w is the longest prefix of w that does not contain the letter B. Let $a_{n,k}$ denote the number of words $w \in \mathcal{W}_n$ with initial A-run of length k. Find a closed-form expression for the generating function $F(x, u) = \sum_{n,k\geq 0} a_{n,k} x^n u^k$. What is the asymptotic probability distribution of the random variables X_n defined as the length of the initial A-run in a uniformly random word $w \in \mathcal{W}_n$? [2 points]

Question 2. Let g_n be the number of 2-regular undirected graphs on the vertex set $[n]$ (labelled, without loops and multiple edges). What is the best asymptotic estimate for g_n you can derive? [2 points]

Question 3. An *involution* of order n is a permutation π of $[n]$ satisfying $\pi(\pi(i)) = i$ for every $i \in [n]$. Let I_n be the number of involutions of order n. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor ! \leq I_n \leq n! e^{\sqrt{n}} \left(\frac{e}{n}\right)^{n/2}. \text{ [2 points]}$$

Can you improve any of these bounds? [2 more points]

Question 4. Let $\mathcal{T}_n$ denote the class of planted binary trees with n leaves, that is, rooted trees in which every internal node has two children, one of which is specified as the left child and the other as the right child. The *depth* of a vertex v in a rooted tree is defined as the number of edges on the path from the root to v. The *total leaf depth* of a tree T, or $\mathrm{tld}(T)$, is the sum of the depths of all the leaves of T. For example, among the five trees in $\mathcal{T}_4$, one has $\mathrm{tld} = 8$ and four have $\mathrm{tld} = 9$. Show that for a uniformly random tree $T \in \mathcal{T}_n$, the expectation of $\mathrm{tld}(T)$ is of order $\Theta(n^{3/2})$. [3 points] (Hint: you may proceed by showing that the corresponding bivariate generating function $F(x, u)$ admits a functional equation involving the function $F(ux, u)$, and use this equation to determine $\frac{\partial F(x,u)}{\partial u}|_{u=1}$.)

Question 5. Let E_n denote the number of permutations of $[n]$ whose every cycle has even length, and let O_n denote the number of permutations of $[n]$ whose every cycle has odd length. Determine

$$\lim_{n\to\infty} \frac{E_{2n}}{O_{2n}}. \text{ [2 points]}$$

Question 6. Let us consider directed graphs without loops in which each ordered pair (u, v) of vertices is connected by at most one directed edge $\overrightarrow{uv}$ (we do allow a directed cycle of length 2, i.e., a pair of edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$). A directed graph $\vec{G}$ is *weakly connected*, if the undirected graph obtained by replacing all directed edges of $\vec{G}$ by undirected edges is connected. A directed graph is *out-d-regular*, if each vertex has exactly d outgoing edges. Explain how to efficiently compute, for a given n and d, the exact number of weakly connected out-d-regular directed graphs on the vertex set $[n]$. [1 point] Can you implement your solution and compute the number of weakly connected out-42-regular graphs on 500 vertices? (Your implementation may use any mathematical software package of your choice.) [2 more points]