NMAG403 - Combinatorics

December 15, 2023 – Diverse Topics, Last Set of Problems

Homework

Deadline: **January 8, 2024**
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1. Let G be a cubic graph with a Hamiltonian cycle. Prove that G has at least 3 Hamiltonian cycles.
2. Let q be a prime and n, d positive integers. Consider $V = GF(q)^n$ as a vector space over $GF(q)$ and let $\mathcal{B}$ consist of all d-dimensional subspaces of V. Show that $(V, \mathcal{B})$ is a (v, k, λ)-BIBD for certain parameters v, k, λ and determine the values of these parameters (as functions of q, n and d).
3. For a graph G, let $\zeta(G)$ be the smallest integer k such that G has an $\mathcal{L}$-list coloring for every list assignment $\mathcal{L} = \{L(u), u \in V(G)\}$ satisfying

   (a) $|L(u)| = k$ for every $u \in V(G)$, and

   (b) $|L(u) \cap L(v)| \leq 1$ for every edge $uv \in E(G)$.

   Prove the following:

   (a) For every $n \geq 3$, $\zeta(K_n) \leq \lfloor \sqrt{n - \frac{11}{4}} + \frac{3}{2} \rfloor$.

   (b) There are infinitely many values of n such that $\zeta(K_n) > \sqrt{n}$.

In class problems

40. Let G be a graph on n vertices and let $\bar{G}$ be its complement (i.e. uv is an edge of $\bar{G}$ if and only if it is not an edge of G). Prove that $ch(G) + ch(\bar{G}) \leq n + 1$, where ch denotes the chooseability.
41. A spanning subgraph H (not necessarily induced) of a graph G is called a *k-factor* of G if $\deg_H u = k$ for every $u \in V(H) = V(G)$. Prove the following: If k is a positive integer and a $2k$-regular graph G has an even number of edges, then G contains a k-factor. Show that the assumption on the parity of the number of edges of G is necessary.
42. A spanning subgraph H (not necessarily induced) of a graph G is called a *k-factor* of G if $\deg_H u = k$ for every $u \in V(H) = V(G)$. Prove the following: If k is a positive integer, then the edges of any $2k$-regular graph G can be partitioned into k disjoint 2-factors of G.
43. Recall the axioms of projective planes from the lecture and prove that if (A1) and (A2) hold, then (A3) is equivalent to (A3').
44. Consider bipartite graphs with both classes of bipartition having the same number n of vertices and which do not contain cycles of length 4.

    (a) Prove that the number of edges in such a graph is at nost $(1 + o(1))n\sqrt{n}$.

    (b) Show that there are infinitely many values n for which there exist such graphs with at least $n\sqrt{n}$ edges.

45. A *finite affine plane* is a set system $\mathcal{A} = (X, \mathcal{L})$ satisfying the following axioms:

A1 For every $L \in \mathcal{L}$ and every $x \in X \setminus L$ there is a unique $L' \in \mathcal{L}$ such that $x \in L'$ and $L' \cap L = \emptyset$.

A2 For every two distinct $x, y \in X$, there is a unique $L \in \mathcal{L}$ such that $x, y \in L$.

A3 For every $L \in \mathcal{L}$ it holds that $|L| \geq 2$.

A4 There exist distinct points $x, y, z \in X$ such that $\{x, y, z\} \nsubseteq L$ for every $L \in \mathcal{L}$ (i.e., x, y, z are not colinear).

(a) Prove that if one removes a line and all of its points from a projective plane, one gets an affine plane.

(b) Prove that if $\mathcal{A}$ is an affine plane, then one can add some new points and one new line (and extend the existing lines by some of the new points) to get a projective plane.

46. Prove that a plane triangulation with more than 3 vertices cannot contain exactly one Hamiltonian cycle. Hint:

(a) Let $G = (V, E)$ be such a graph and let $H = x_1, x_2, \ldots, x_n$ be a Hamiltonian cycle in G. For every edge e of H, let $M(e)$ be the two triangular faces incident with e. Prove that M has a system of distinct representatives s.

(b) Let F be the domain of s (i.e., the assigned faces) and consider the bipartite graph G' with one partition V, the other partition F and vertex $v \in V$ connected to a face $f \in F$ if and only if $v \in f$. Observe that each Hamiltonian cycle in G' gives a Hamiltonian cycle in G.

(c) Prove that G' has at least two Hamiltonian cycles. If at least one of them does not correspond to H, we are done, otherwise do some extra work to prove the main statement anyway.