NMAG403 - Combinatorics

November 10, 2023 – Chromatic number and Chooseability II

In class problems

31. Let $G = (A \cup B, E)$ be a bipartitie graph with A and B being its classes of bipartition. Let Δ be a positive integer such that $\Delta(G) \leq \Delta$. Fix a proper edge-coloring $\varphi : E \to \{1, \dots, \Delta\}$ (we know from homework 1.2 that such an edge-coloring always exists). Consider the line graph $L(G)$ of G and define an orientation $\to$ as follows: If edges $e, f \in E$ share a vertex u, then the edge ef of the line graph $L(G)$ is oriented from e to f if $\varphi(e) < \varphi(f)$ and $u \in A$, or if $\varphi(e) > \varphi(f)$ and $u \in B$.

    (a) Show that for every subgraph H of G, the line graph $L(H)$ of H has a kernel with respect to the orientation induced on $L(H)$ by the orientation $\to$.

    (b) Observe that the outdegree of any vertex of $\overrightarrow{L(G))}$ does not exceed $\Delta - 1$.

    (c) Deduce from these that $ch(L(G)) = \Delta(G)$.

32. Denote by $g(G)$ the smallest Euler genus g of a surface S_g that allows a non-crossing embedding of G in S_g.

    (a) Show that $g(K_7) = 1$.

    (b) Determine $g(K_8)$.

33. With the help of the Four Color Theorem, prove that every bridgeless planar 3-regular graph is edge-3-colorable.

34. Show that for every (orientable) surface, there are only finitely many mutually non-isomorphic connected 7-regular graphs which allow a non-crossing drawing on that surface.

35. Prove that for every $g > 0$, every graph of genus at most g is $H(g)$-choosable (here $H(g) = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$ is the Heawood number).