NMAG403 - Combinatorics

October 13, 2023 – Hall's theorem

Homework

Deadline: **November 13, 2023**
Send to: `honza@kam.mff.cuni.cz` (in PDF)

1. (a) Does the system of all 3-element subsets of $\{1,2,3,4\}$ have an SDR?
   (b) Does the system of all 3-element subsets of $\{1,2,3,4,5\}$ have an SDR?
   (c) Let $B_{n,k}$ be the bipartite incidence graph of the system of all k-element subsets of an n-element set. What is $\mu(B_{n,k})$? (Give a closed formula and prove it.)
2. Prove that $\chi'(G) = \Delta(G)$ holds true for every bipartite graph G. (Here $\Delta(G)$ denotes the maximum degree of a vertex of G and $\chi'(G)$ denotes the edge chromatic number, aka chromatic index, of G.)
3. For which k is the following statement true? Every legal filling of the first k lines of a SUDOKU can be extended to a legal completion of the entire 9×9 table. Prove your answer.

In class problems

9. Prove or find a counterexample to the following statement: Let I and X be **infinite** sets and let $\mathcal{M} = \{M_i\}_{i \in I}$ be a set system such that $\bigcup \mathcal{M} = X$. If $\mathcal{M}$ satisfies the Hall condition ($\forall J \subseteq I : |\bigcup_{j \in J} M_j| \geq |J|$), then $\mathcal{M}$ has an SDR.
10. Let G be a bipartite graph with 42 vertices such that whenever you pick 31 vertices, they will contain at least one edge. Show that G has a matching with at least 12 edges.
11. Prove, for every integer k, that a graph has an orientation of maximum outdegree at most k if and only if each of its subgraphs H satisfies $|E(H)| \leq k \cdot |V(H)|$.
12. Dilworth's theorem says that if a finite poset $(P, \prec)$ has the largest antichain of size r, then P can be decomposed into r chains. Prove that Dilworth's theorem implies the harder implication of Hall's theorem.
13. Prove Birkhoff's theorem which says that for every n, the set of all bistochastic matrices of order n is exactly the convex hull of the set of permutation matrices of the same order (matrices being viewed as points in n^2-dimensional Euclidean space).